

Mathematics in the News

Shizuo Kakutani

A mathematician for the ages

A drunk man will find his way home, but a drunk bird may get lost forever ^[1]



From L to R: S. Alpern, A. Hajian, M. Stone, A. Bellow, and S. Kakutani in Amherst, MA

Shizuo Kakutani

Shizuo Kakutani, mathematics professor at Yale University and one of the great mathematicians of the last century, died at the age of 92, this past August in Hamden, Connecticut. He is survived by his wife Kay and daughter Michiko.

Widely known to most mathematicians, he has proved fundamental results in complex analysis, topological groups, functional analysis, probability theory, Brownian motion, topology, and ergodic theory. Amongst economists and non mathematicians he is probably most famous for his fixed point theorem (*A generalization of Brouwer's fixed point theorem* (Duke Math J.,1941)), a result we explain in this note: Kakutani's fixed point theorem is extensively used by mathematical economists (a 1982 survey by G. Debreu cites over 350 instances where Kakutani's fixed point theorem is the main tool for proving the existence of an economic equilibrium). This fixed point theorem is the key step in the work of three researchers who received Nobel Prizes in economics: John Nash (the subject of Sylvia Nassar's book *A Beautiful Mind*) used it in his original 1950

proof of the existence of Nash equilibria, work for which he received the Economics Nobel almost 40 years later in the 1990's. In 1954, the economists Arrow and Debreu used Kakutani's fixed point theorem to prove there are prices for goods that balance supply and demand in a complex economy. Arrow and Debreu's work was awarded the Economics Nobel prize in 1972.

Fixed point theorems

Consider the following fixed point problems in different dimensions:

In dimension 1 (you may recall your Intermediate Value Theorem from Calculus I days)

One morning, exactly at sunrise, a monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed. Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day. [\[2\]](#)

Brouwer's fixed point theorem in dimension 2:

Take two pieces of 8 by 11 paper and lay them on top of one another so that every point on the top paper corresponds with a point on the bottom paper. Now crumple the top piece of paper in any way that you wish (without tearing) and place it back on top.

Brouwer's theorem tells us that there must be a point which has not moved, i.e. which lies exactly above the same point that it did initially.

Brouwer's fixed point theorem in dimension 3:

You have a cup of coffee in front of you to which you give a quick stir. After the coffee has stopped moving, Brouwer's theorem tells you that there is a point which is in its original spot before the stirring.

Brouwer's theorem in dimension n (guide: think of S as our coffee and for each point of coffee x , $f(x)$ is where x has moved to after stirring).

If $x \rightarrow f(x)$ is a continuous point-to-point mapping of an n -dimensional closed simplex S into itself then there is a point x_0 such that $x_0 = f(x_0)$ (i.e., every continuous stirring leaves some point x_0 in its original location).

Kakutani's fixed point theorem, generalizing Brouwer's result:

Let S be an n -dimensional closed simplex and consider $C(S)$ the family of all nonempty closed convex subsets of S . A point-to-set mapping $x \rightarrow F(x) \in C(S)$ of S into $C(S)$ is called upper semi-continuous if whenever $x_i \rightarrow x$ and $y_i \in F(x_i)$ and $y_i \rightarrow y$ then $y \in F(x)$. A point-to-set map F is sometimes called a correspondence.

If $x \rightarrow F(x)$ is an upper semi-continuous point-to-set mapping of an n -dimensional closed simplex into $C(S)$, then there is a point x_* such that $x_* \in F(x_*)$. We call x_* a fixed point for the correspondence F .

The proof is a typical Kakutani argument: clear, concise and aesthetic. He applies Brouwer's fixed point theorem to a sequence of point-to-point maps f_k to produce a sequence of points x_k (the f_k are related to F , via the k 'th barycentric subdivision of the simplex, to get technical).

The sequence x_k has a limit point x_* which is the required fixed point. Furthermore this theorem is valid even if S is an arbitrary closed bounded convex set in Euclidean space. Brouwer's theorem is a special case when each $F(x)$ consists of only one point $f(x)$. In this case the upper semi-continuity of F is nothing but the continuity of f .

Personal recollection (Raj Prasad)

Much of what I have written above can be found on the web. For more details about Kakutani's life, follow the url <http://faculty.uml.edu/vprasad/KakutaniTangents.pdf>. I was fortunate, at the start of my career in the late 70's, to work with Kakutani when I was a postdoctoral fellow at Yale. Kakutani was a warm and gracious person who was genuinely concerned about his students and colleagues. Contrary to the public perception that mathematical wizards are eccentric or crazy, Kakutani was a level headed and delightful person to be around. His seminars and classes were exciting events in which to participate - I saw many theses come from these sessions. Friends and former students of Shizuo, would regularly meet with him in New Haven or nearby. The photo in this article is in fact from a summer meeting in Amherst, Massachusetts. After having lunch we would talk mathematics for the afternoon, and then all head back separately to our homes. In the picture above, from left to right are S. Alpern of LSE (back turned, and one of my favorite collaborators), A. Hajian of Northeastern (former student of Kakutani's), Marshall Stone of Harvard, Alexandra Bellow of Northwestern (another ex-student of Kakutani) and of course Shizuo Kakutani. Sessions with Kakutani were always fun, whether it involved discussing mathematics, mathematicians or if the Red Sox would ever finally win the World Series.

[1] A joke by Shizuo Kakutani at a UCLA colloquium talk as attributed in Rick Durrett's book *Probability: Theory and Examples*. Kakutani's joke is an application of Polya's result that a simple random walk is recurrent in

dimensions 1 or 2, but transient in 3 dimensions or higher.

[2] Imagine there are two monks, one going down and one going up, each beginning on the same day at sunrise. At some the point in the day, the two monks must meet!