Fourier Series Tutorial

INTRODUCTION

This document is designed to overview the theory behind the Fourier series and its applications. It introduces the Fourier series and then demonstrates its use with a detailed example. The Fourier Series MATLAB GUI can be used to follow along with the example shown here.

BASIC CONCEPTS

A Fourier series is a method of representing a complex periodic signal using simpler signals. These simple signals are sinusoids which are summed to produce an approximation of the original signal. The approximation becomes more accurate as more terms are used. The basic form of a Fourier series is

$$x(t) = a_0 + a_1 \cos(\omega_0 t + \theta_1) + a_2 \cos(2\omega_0 t + \theta_2) + \ldots + a_N \cos(N\omega_0 t + \theta_N), \tag{1}$$

where
$x(t)$ = the approximation of the original signal,
a_0 = a constant, which produces a “DC offset,”
$a_1, a_2, \ldots a_N$ = constant terms which change the amplitude of the sinusoidal signals,
ω_0 = the dominant frequency of the signal, and
$\theta_1, \theta_2, \ldots \theta_N$ = phase shifts.

This form shown with only cosines, but sines or a combination of sines and cosines can be used.

APPROXIMATING A SQUARE WAVE

Fourier series will be demonstrated by showing how to approximate a square wave, such as that shown in Fig. 1, using three sine waves.

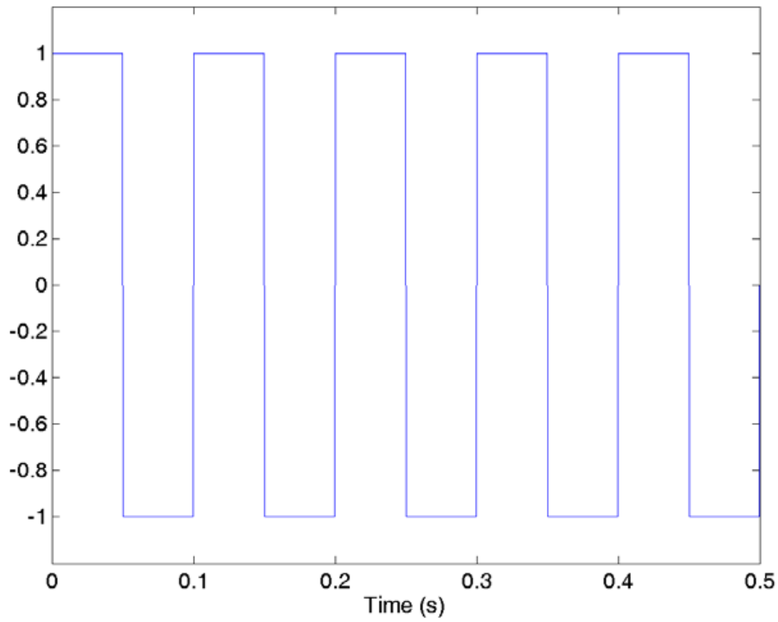

Fig. 1. Square wave.

Sine waves can be summed, and the resulting signal examined, using a simple SIMULINK model such as the one shown in Fig. 2.

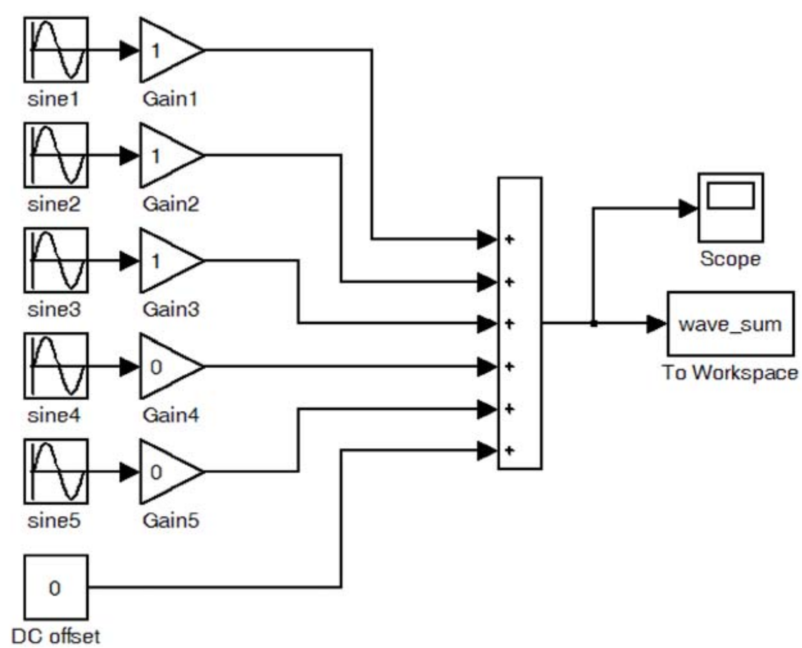

Fig. 2. SIMULINK model to add multiple sine waves.

This model is controlled by the Fourier Series MATLAB GUI. The operation of the GUI is fairly straightforward, but for more details see the Fourier Series GUI documentation. This model can sum up to five sine waves, but for the sake of simplicity only three will be used for this demonstration.

If three sine waves with the appropriate amplitudes and frequencies are summed and plotted, we see that the resulting plot resembles the original square wave, as seen in Fig. 3.

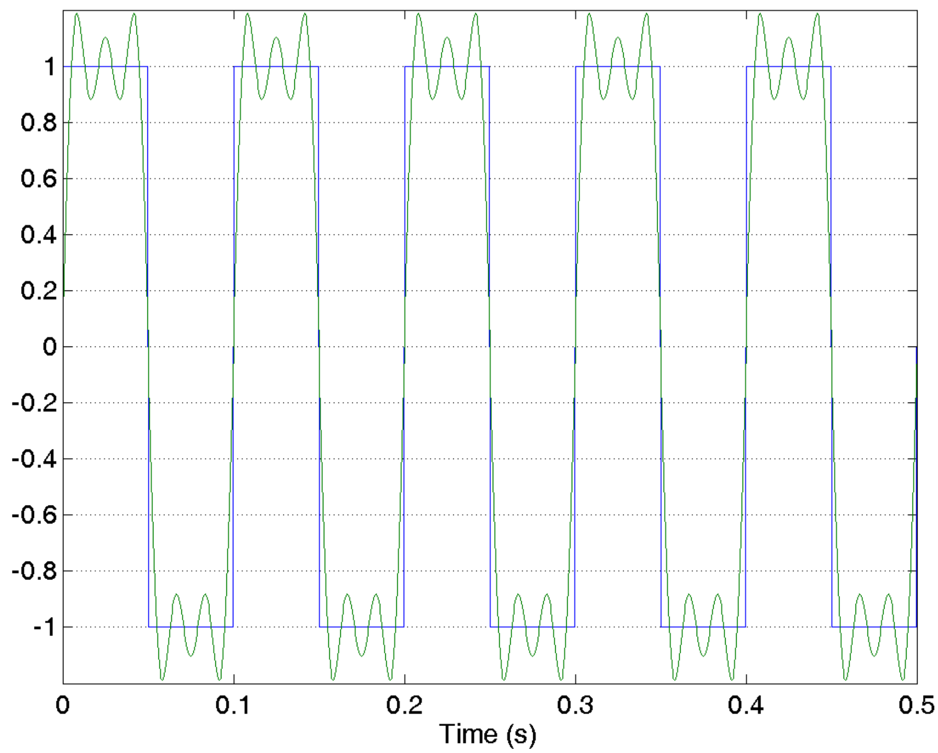

Fig. 3. Square wave approximation.

If the Fourier series were summed from 0 to infinity (an infinite number of terms), the result would be an exact square wave. Using the Fourier Series GUI, try adjusting the frequencies and magnitudes of three sine waves to approximate the square wave (hint: do not adjust the phase).

This is difficult to do without knowing the correct values to use. The following information about the Fourier Series will help you choose the correct values for the sine waves.

Determination of constants

To approximate a particular signal using a Fourier series, the correct constants must be determined. The Fourier series can be written as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{p} + b_n \sin\frac{2n\pi t}{p}\right), \tag{2}$$

where a_0, a_n, and b_n are constants, and n = 1, 2, 3, etc. Note that here the series has been written with both sine and cosine terms, rather than using only phase-shifted cosine terms as in (1). The constant terms a_0, a_n, and b_n are calculated using

$$a_0 = \frac{2}{p}\int_{-p/2}^{p/2} f(t)\,dt\,, \tag{3}$$

$$a_n = \frac{2}{p}\int_{-p/2}^{p/2} f(t)\cos\frac{2n\pi t}{p}\,dt \quad \text{for } n = 1,2,3,\ldots,\text{ and} \tag{4}$$

$$b_n = \frac{2}{p}\int_{-p/2}^{p/2} f(t)\sin\frac{2n\pi t}{p}\,dt \quad \text{for } n = 1,2,3,\ldots, \tag{5}$$

where
f(t) = the periodic function to be approximated, and
p = the period of the function.

We will now calculate a_0, a_n, and b_n for the square wave shown in Fig. 1. The analytical description of the function is

$$f(t) = \begin{cases} -1 & \text{if } -0.05 < x < 0 \\ 1 & \text{if } 0 < x < 0.05 \end{cases} \tag{6}$$

and the period (p) is 0.1 seconds. Therefore

$$a_0 = \frac{2}{p}\int_{-p/2}^{p/2} f(t)\,dt = \frac{2}{0.1}\left[\int_{-0.05}^{0} -1dt + \int_{0}^{0.05} 1dt\right] = 0, \tag{7}$$

which makes sense because the original signal has no DC bias. Then, the coefficient of the cosine term is

$$\begin{aligned} a_n &= \frac{2}{p}\int_{-p/2}^{p/2} f(t)\cos\frac{2n\pi t}{p}\,dt = \frac{2}{0.1}\left[\int_{-0.05}^{0} -1\cos\frac{2n\pi t}{0.1}\,dt + \int_{0}^{0.05} 1\cos\frac{2n\pi t}{0.1}\,dt\right] \\ &= \frac{1}{n\pi}\left[-\sin(20n\pi t)\Big|_{-0.05}^{0} + \sin(20n\pi t)\Big|_{0}^{0.05}\right] = 0, \end{aligned} \tag{8}$$

for n = 1, 2, 3,…, and the coefficient of the sine term is

$$\begin{aligned} b_n &= \frac{2}{p}\int_{-p/2}^{p/2} f(t)\sin\frac{2n\pi t}{p}\,dt = \frac{2}{0.1}\left[\int_{-0.05}^{0} -1\sin\frac{2n\pi t}{0.1}\,dt + \int_{0}^{0.05} \sin\frac{2n\pi t}{0.1}\,dt\right] \\ &= \frac{2}{n\pi}[1-\cos(n\pi)] = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{if n is even} \\ 4/(n\pi) & \text{if n is odd} \end{cases} \end{aligned} \tag{9}$$

For a square wave, as for all odd functions, the coefficient of the sine term (a_n) vanishes. For even functions the coefficient of the cosine term (b_n) vanishes. To review, an even function is one where

$$f(x) = f(-x), \tag{10}$$

so the function is symmetrical about the y-axis. For an odd function,

$$f(x) = -f(-x). \tag{11}$$

In addition, the Fourier series for a square wave only has non-zero terms when n is an odd number. The equation which describes our square wave is then

$$f(t) = \sum_{n=1,3,5,\ldots}^{\infty}\left[\frac{4}{n\pi}\sin(2\pi f nt)\right]. \tag{12}$$

Plugging in the frequency of the square wave, 10 Hz, the first three terms of the series are

$$f(t) = \frac{4}{\pi}\left[\sin(20\pi t) + \frac{1}{3}\sin(60\pi t) + \frac{1}{5}\sin(100\pi t)\right]. \tag{13}$$

For the sake of putting these values in the GUI, the amplitudes and frequencies of the first three sine waves are given in decimal form in Table 1.

Table 1: Amplitude and frequency of first three terms.

Term	Frequency	Amplitude
1	10 Hz	1.273
2	30 Hz	0.424
3	50 Hz	0.255

Go back to the Fourier Series GUI and insert these values; the result resembles a square wave.

The general trend of a summation of sine waves can be predicted by looking at the “slowest” sine wave. The signal with the lowest frequency, determines the general shape of the sum. In the previous example, we saw that the plot approximated a 10 Hz sine wave. The lowest frequency was 10 Hz. Test this idea using the GUI, and also try to replicate the other sample signals in the GUI.