MODAL SPACE - IN OUR OWN LITTLE WORLD by Pete Avitabile

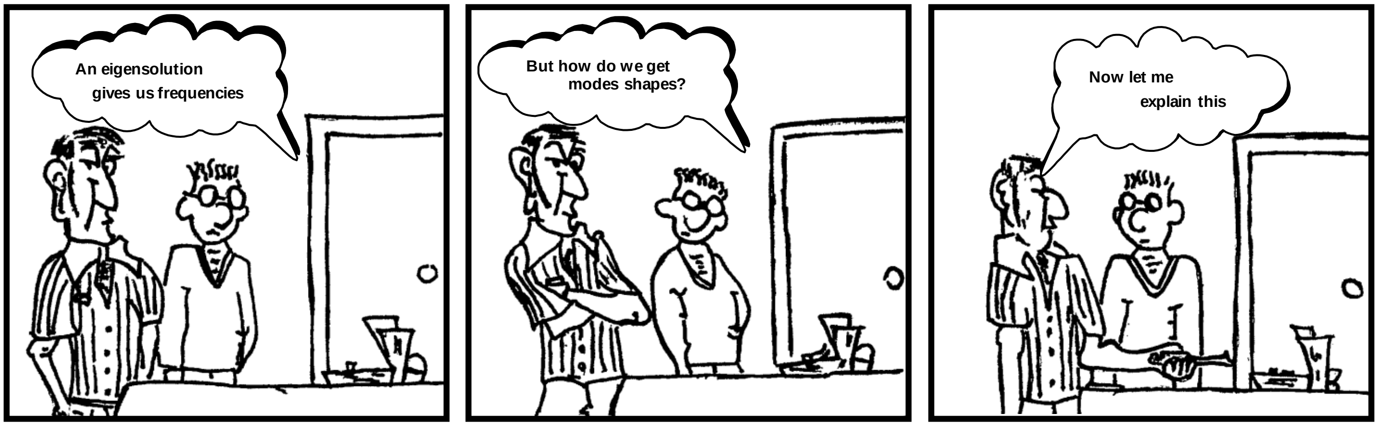

Illustration by Mike Avitabile

An eigensolution gives us frequencies. But how do we get mode shapes?
Now let me explain this.

OK. I think the first thing that we have to say is that the eigensolution actually gets us both the frequencies and mode shapes. The mathematical process of the eigensolution can be performed a number of different ways. There are what are called direct techniques and indirect techniques for the solution.

For smaller matrices, the direct techniques decompose the set of equations to get all of the eigenvalues and eigenvectors. Techniques such as Jacobi, Givens and Householder are common methods that are used.

But when the matrices get larger, like those of the large finite element model that are generally developed today, then an indirect technique is used where only a few of the lower order modes are obtained. Techniques such as Subspace Iteration, Simultaneous Vector Iteration and Lanczos are some of these indirect techniques that are used.

But I really don't want to make this article a math class or really get into the details of the solution sequence. So let's discuss the eigensolution and what we are attempting to do when we find the frequencies and mode shapes. I want to explain it so it makes sense to you.

So let's write the eigensolution in general form.

$$[[K]-\lambda[M]]\{x\}=\{0\} \tag{1}$$

The first thing that I want to say is that the eigenvalues can be found from the determinant of the matrix. Well, that determinant will really be nothing more than a high order polynomial whose roots are the eigenvalues. Now numerically those can be obtained from any root solving algorithm such as Secant Method or Newton-Rapson Method as a few well known approaches.

So the eigen equation and a typical polynomial that may result is shown in Figure 1. The function zero crossings are the roots where the polynomial is zero.

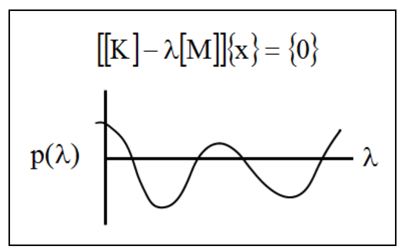

Figure 1: Graphical Representation of Roots of Determinant

Now that gives us the frequencies of the set of equations and the next step is to determine the mode shapes. Well, if you take the first eigenvalue, $\lambda = \omega_1^2$, and substitute it into the eigensolution equation, then you can solve for the $\{x_1\}$ vector because you know [M], [K], and ω_1^2. The solution for that vector is straightforward using any decomposition scheme such as Crout-Doolittle, Cholesky, LDL decomposition to name several well known popular approaches.

So that $\{x_1\}$ vector is actually the mode shape for that particular frequency that was used to solve the set of equations. Figure 2 shows this schematically for the first free-free mode for a simple beam; note that blue is used to identify this as the first mode of the system. And if you follow through with the equation in Figure 2 you will notice that the elastic forces are equal to the inertial forces in the way they are written. We could also say the the beam is in dynamic equilibrium at that frequency which is ω_1^2 . And if you looked at the system from an energy perspective you could see why there are node points where the system oscillates about those points and there is equal positive and negative parts of the shape to keep it in equilibrium.

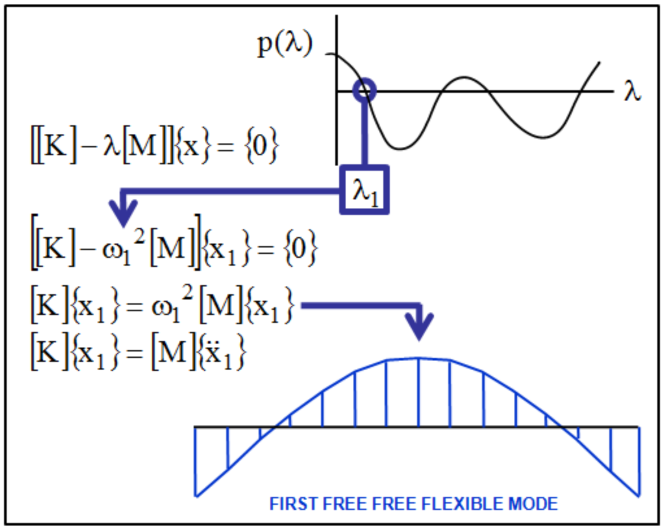

Figure 2: Schematic for eigensolution for mode 1

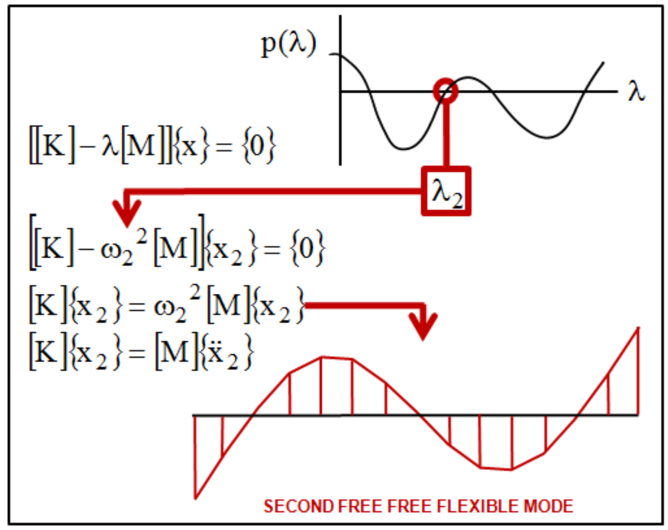

Figure 3: Schematic for eigensolution for mode 2

Of course now we need to do the same thing for the second frequency. If you now take the second eigenvalue, $\lambda = \omega_2^2$, and substitute it into the eigensolution equation, then you can solve for the $\{x_2\}$ vector because you know [M], [K], and ω_2^2. Now the $\{x_2\}$ vector is actually the mode shape for the second frequency. Figure 3 shows this schematically for the second free-free mode for a simple beam; note that red is now used to identify this as the second mode of the system. Again let's follow through with the equation in Figure 3 in red you will notice that the elastic forces are equal to the inertial forces in the way they are written. We could also say the the beam is in dynamic equilibrium but now at the frequency which is ω_2^2. And just like we did with mode 1, we will see that the node points are locations where the system oscillates about those points and there is equal positive and negative parts of the shape to keep it in equilibrium.

We then continue this process for all the modes of interest. Of course the way I explained it may not be the way the different solution algorithms actually decompose the matrices and obtain the final answer. But the way I have explained it will probably give you a much better overall idea how the frequencies and mode shapes come from the system set of equations.

So it is important to realize that the eigensolution is used to obtain what is called the eigenpair – that is, the frequency and the vector associated with the eigen-equation. This is in fact the mode shape.

Now another thing to realize is that the mode shapes are linearly independent and the mode shapes are also orthogonal with respect to the mass and stiffness matrices. This is a by-product of the eigensolution. This is a very important fact that is often used when we check our finite element model with measured experimental data. We perform a type of orthogonality check, often called a pseudo-orthogonality check, to compare the measured experimental vectors with those obtained from the eigensolution.

I hope that this helps to explain the questions you had. If you have any other questions about modal analysis, just ask me.