MODAL SPACE - IN OUR OWN LITTLE WORLD

by Pete Avitabile

Illustration by Mike Avitabile

Can you describe reciprocity? It just doesn't make sense to me.
This is something that often confuses people.

Alright – let's discuss the reciprocity of measurements when doing modal testing. This is a very important item when doing modal tests. People say the words but sometimes they really don't believe it – mainly because when we take measurements there are many reasons why the actual measurement may not satisfy the theoretical requirement of reciprocity.

Let's first simply state what reciprocity is. Figure 1 shows a structure where an input-output measurement is to be made at point "i" and point "j". Now in one measurement the force is applied at point "i" and the response is measured at point "j". And in the second measurement, the force is measured at point "j" and the response is measured at point "i". From the principle of reciprocity, the h_{ij} must equal h_{ji}

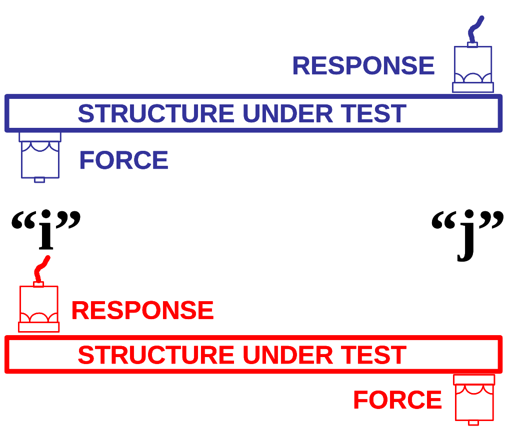

Figure 1 – Schematic for Reciprocity Measurement

From the complete set of measurements possible, Figure 2 shows a frequency response matrix where one row and one column are measured. Several reciprocal measurements are highlighted in that matrix for reference.

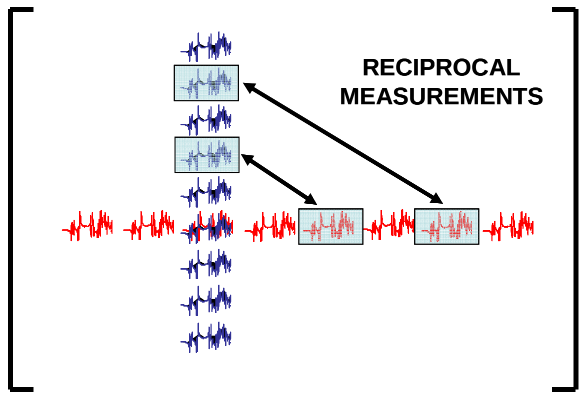

Figure 2 – FRF Matrix Showing Reciprocal Measurements

Of course, the first time you try to explain reciprocity to someone who is not familiar with this concept, it always seems to raise an eyebrow. So let's try to show where reciprocity comes from in the basic equations describing the system. (Some theory will have to be presented here to show reciprocity)

First let's realize that we start from an equation of motion written in matrix form for a multiple degree of freedom system as:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$

Now the important point to note here is that these matrices are square symmetric (for a structural system). This immediately implies that the "ij" and "ji" terms of the matrix are the same.

Now let's write the equation of motion in the Laplace domain for that physical equation of motion written above. Assuming initial conditions are zero, then this is:

$$\left[[M]s^2+[C]s+[K]\right]\{X(s)\}=\{F(s)\}$$

Of course we have to realize that each of the terms of this matrix is also square symmetric. From this Laplace equation of motion, the system matrix [B(s)] and its inverse, the system transfer function [H(s)], is also square symmetric. This is:

$$\left[B(s)\right]^{-1} = \left[H(s)\right] = \frac{\mathrm{Adj}\left[B(s)\right]}{\det\left[B(s)\right]} = \frac{\left[A(s)\right]}{\det\left[B(s)\right]}$$

Now with some manipulation, the system transfer function can be written in partial fraction form as the summation of all the individual modes of the system. This is:

$$\left[H(s)\right] = \sum_{k=1}^{m} \frac{\left[A_k\right]}{\left(s - p_k\right)} + \frac{\left[A_k^*\right]}{\left(s - p_k^*\right)}$$

The frequency response function is the system transfer function evaluated at s-jω and is given as:

$$\left[H(s)\right]_{s=j\omega} = \left[H(j\omega)\right] = \sum_{k=1}^{m} \frac{\left[A_k\right]}{\left(j\omega - p_k\right)} + \frac{\left[A_k^*\right]}{\left(j\omega - p_k^*\right)}$$

Now it is important to remember that the residue matrix [A(s)] is also square symmetric because all the matrices used to ultimately form it were square symmetric.

Now a single "ij" measurement can be written as:

$$h(s)_{ij}\Big|_{s=j\omega} = h_{ij}(j\omega) = \sum_{k=1}^{m} \frac{a_{ijk}}{(j\omega - p_k)} + \frac{a_{ijk}^*}{(j\omega - p_k^*)}$$

and expanding the first three mode terms for this gives:

$$\begin{aligned} h_{ij}(j\omega) = {} & \frac{a_{ij1}}{(j\omega - p_1)} + \frac{a_{ij1}^*}{(j\omega - p_1^*)} \\ & + \frac{a_{ij2}}{(j\omega - p_2)} + \frac{a_{ij2}^*}{(j\omega - p_2^*)} \\ & + \frac{a_{ij3}}{(j\omega - p_3)} + \frac{a_{ij3}^*}{(j\omega - p_3^*)} \end{aligned}$$

But in this form it is not clearly obvious that reciprocity exists. So the residue form of the equation needs to be extended. Recall that the residue matrix for the kth mode of the system can be obtained from singular value decomposition and written as:

$$\left[A(s)\right]_k = q_k \{u_k\}\{u_k\}^T$$

or expanded as:

$$\begin{bmatrix} a_{11k} & a_{12k} & a_{13k} & \cdots \\ a_{21k} & a_{22k} & a_{23k} & \cdots \\ a_{31k} & a_{32k} & a_{33k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = q_k \begin{bmatrix} u_{1k}u_{1k} & u_{1k}u_{2k} & u_{1k}u_{3k} & \cdots \\ u_{2k}u_{1k} & u_{2k}u_{2k} & u_{2k}u_{3k} & \cdots \\ u_{3k}u_{1k} & u_{3k}u_{2k} & u_{3k}u_{3k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and in this form is simply written as:

$$h(s)_{ij}\Big|_{s=j\omega} = h_{ij}(j\omega) = \sum_{k=1}^{m} \frac{q_k u_{ik} u_{ij}}{(j\omega - p_k)} + \frac{q_k^* u_{ik}^* u_{jk}^*}{(j\omega - p_k^*)}$$

and can be expanded for the first three mode terms as:

$$\begin{aligned} h_{ij}(j\omega) = {} & \frac{q_1 u_{i1} u_{j1}}{(j\omega - p_1)} + \frac{q_1 u_{i1} u_{j1}{}^*}{(j\omega - p_1^*)} \\ & + \frac{q_2 u_{i2} u_{j2}}{(j\omega - p_2)} + \frac{q_2 u_{i2} u_{j2}{}^*}{(j\omega - p_2^*)} \\ & + \frac{q_3 u_{i3} u_{j3}}{(j\omega - p_3)} + \frac{q_3 u_{i3} u_{j3}{}^*}{(j\omega - p_3^*)} \end{aligned}$$

Now in this form the reciprocity can be very easily seen. This is because the residue is nothing more than the value of the mode shape at the ith degree of freedom times the value of the mode shape at the jth degree of freedom (plus a few other terms that are constant). This implies that it doesn't matter whether we measure force at point "i" or point "j" when we measure the response at the other point – the product of the mode shape values at the input and output location is the same. So reciprocity must happen for this case. As an example, a simple 3x3 FRF matrix is shown in Figure 3 for magnitude plots. Clearly, the reciprocity can be seen in the matrix. Note that the real, imaginary and phase are also symmetric but not shown here for brevity.

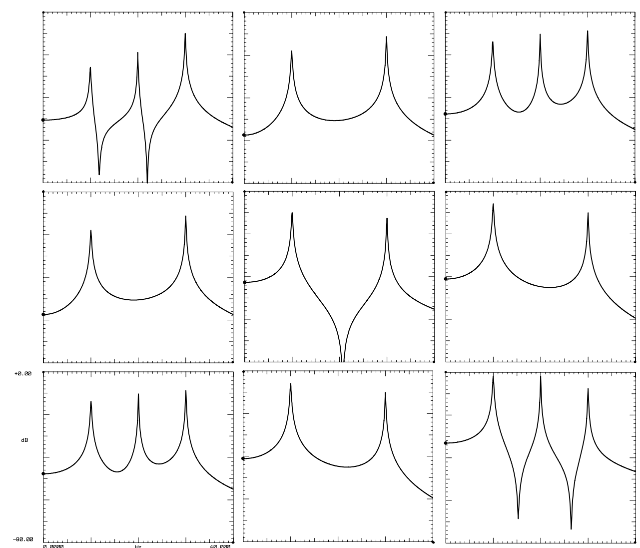

Figure 3 – Magnitude FRF Matrix

Of course this is a theoretical presentation and must hold true. However, measurements may not be so cooperative all the time. This will be discussed at some future point in time.

I hope that this discussion clears up any confusion as to why reciprocity must hold true. While there were some theoretical equations presented, they are necessary in order to show that reciprocity must hold true. If you have any more questions on modal analysis, just ask me.