MODAL SPACE - IN OUR OWN LITTLE WORLD

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I hear about SVD all the time
Could you explain it simply to me?
Sure ...

I'm surprised you haven't asked this question sooner. SVD, singular valued decomposition, is probably one of the most important linear algebra tools that we use today to solve many of our structural dynamic problems. First let's present the mathematical formulation of SVD and some of its variations and then describe where it is commonly used in experimental modal analysis. Of course, I will try to explain the use of SVD and its use rather than give a detailed mathematical development.

First we have to realize that we are going to be dealing with matrices here. (I know you all shudder when we say matrices - but as I have said before "Matrices are your friends!".) So let's assume that we have some matrix [A] that is a n x n square matrix. The basic SVD equation is

$$[A]=[U][S][V]^T$$

Now this formulation looks pretty simple but let's expand out some of these terms to see the real power of SVD

$$[A]=\left[\{u_1\} \quad \{u_2\} \quad \{u_3\} \quad \cdots\right]\begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots \end{bmatrix}\left[\{v_1\}^T \quad \{v_2\}^T \quad \{v_3\}^T \quad \cdots\right]$$

The expansion of this gives

$$[A]=\{u_1\}s_1\{v_1\}^T+\{u_2\}s_2\{v_2\}^T+\{u_3\}s_3\{v_3\}^T+\quad\cdots$$

Now that's pretty incredible because it implies that the matrix A is made up of a set of vectors and singular values that describe the matrix.

We could also say that there are parts of the matrix A that are comprised of other matrices who are very simply described as one vector and a corresponding eigenvalue. So the SVD really has the ability to determine the "principal pieces" that comprise the matrix. This also implies that the rank of the matrix can be determined. So let's try a few numbers here to see what this means.

Let's start with a simple vector with an eigenvalue to illustrate the basic SVD equation. Let's define a vector with a singular value as

$$u_1=\begin{Bmatrix}1\\2\\3\end{Bmatrix};\ s_1=1$$

So the matrix A can be found by simply multiplying out these terms to be

$$[A]_1=\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$$

So this is pretty neat because I started with a vector and I formed a matrix. Now this matrix is clearly a 3x3 matrix in size, but what can I say about its rank? Well, if I look at the different rows of the matrix, I can very quickly see that row two and three are linearly related to row 1. That means that while I have a 3x3 matrix, there is only one linearly independent piece of information that makes up this matrix. (Of course, we know that this is true since we made the matrix from one vector). We would then say that this matrix has a rank of 1 - because there is only one linearly independent piece of information that makes up this matrix.

Now let's consider another simple vector with an eigenvalue as

$$u_{2}=\begin{Bmatrix}1\\1\\-1\end{Bmatrix};\ s_2=1$$

So the matrix A can be found by simply multiplying out these terms to be

$$[A]_2=\begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{bmatrix}$$

Again I make all the same comments about this matrix as I did for the first matrix we looked at. The rank of this matrix is 1 because it is made up from one linearly independent piece of information.

Now let's consider a general matrix as

$$[A]_3=\begin{bmatrix}2 & 3 & 2\\3 & 5 & 5\\2 & 5 & 10\end{bmatrix}$$

Now this matrix is a 3x3 but it is not clear to me what its rank is. The simpliest way to determine this is to do an SVD on this matrix. The resulting decomposition is

$$[A]=\begin{bmatrix}\begin{Bmatrix}1\\2\\3\end{Bmatrix} & \begin{Bmatrix}1\\1\\-1\end{Bmatrix} & \begin{Bmatrix}0\\0\\0\end{Bmatrix}\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\begin{bmatrix}\{1 & 2 & 3\}\\\{1 & 1 & -1\}\\\{0 & 0 & 0\}\end{bmatrix}$$

So the beauty of SVD is that I can write the matrix A in terms of the linearly independent pieces that make up the matrix. This can be expressed in summation form as

$$[A]=\begin{Bmatrix}1\\2\\3\end{Bmatrix}1\{1\quad 2\quad 3\}+\begin{Bmatrix}1\\1\\-1\end{Bmatrix}1\{1\quad 1\quad -1\}+\begin{Bmatrix}0\\0\\0\end{Bmatrix}0\{0\quad 0\quad 0\}$$

So I think this helps to explain the basic principles of SVD. But now I need to discuss some of the applications where SVD is commonly used. (There are many different applications for SVD but only a few specific ones related to experimental testing issues are addressed.)

One application of SVD is for the collection of MIMO data for an experimental modal test. While the data acquisition system may generate forcing functions for all of the MIMO shakers that are uncorrelated (linearly unrelated), the actual shaker force excitation may not be completely uncorrelated for each of the shakers due to the interaction of the shakers with the structure.

The linear independence of the input spectrum matrix needs to be checked. During the acquisition of MIMO data, the Gxx matrix of the shakers can be used to perform what is commonly called a principal component analysis. This technique decomposes the Gxx matrix using SVD and then plots the singular values for each of the inputs on a frequency basis. If the shakers are all linearly independent, then there will be a significant singular value at all frequencies for each of the independent inputs. This is shown in Figure 1.

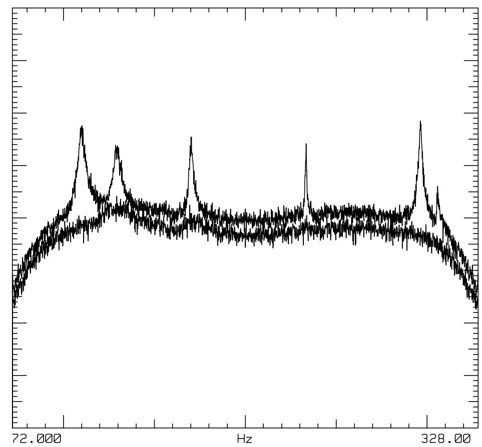

Figure 1 - 2 Shaker MIMO SVD

Another application is used in modal parameter estimation. The FRF matrix from several different references can be decomposed using SVD to determine where there are roots (or modes) of the system. This decomposition is the basis of the CMIF modal parameter estimation approach. A plot of the significant singular values of this SVD will provide plots which will indicate where the modes of the system are located. A typical plots of this is shown in Figure 2 for a system that has repeated roots.

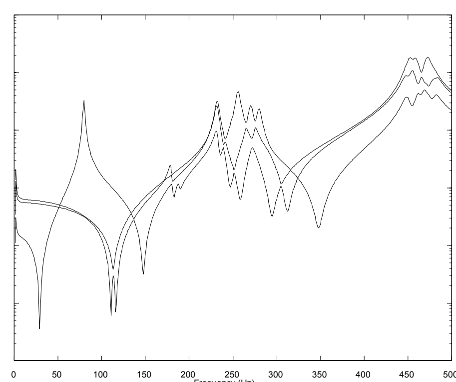

Figure 2

While there are many more applications of SVD, I hope that these few examples help you better understand the technique. If you have any other questions about modal analysis, just ask me.