MODAL SPACE - IN OUR OWN LITTLE WORLD

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What's the difference between a complex mode and a real normal mode?
There's a lot to explain but let's start with some simple examples.

Now that's a question that comes up often and gets many people confused. So let's discuss this in a little detail to explain the differences. Unfortunately, we are going to have to include a little math and some theory here to help explain this.

Let's start with an undamped set of equations and proceed on to a damped case with proportional and then non-proportional damping. It is here where the differences will become apparent. A simple example will be used to illustrate some points here. The equations describing a general system can be written as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}$$

where [M], [C], [K] are the mass, damping and stiffness matrices respectively, along with the corresponding acceleration, velocity, displacement and force.

The transformation to modal space will yield

$$\begin{bmatrix} \diagdown & & \\ & \overline{M} & \\ & & \diagdown \end{bmatrix}\{\ddot{p}\}+\begin{bmatrix} \diagdown & & \\ & \overline{C} & \\ & & \diagdown \end{bmatrix}\{\dot{p}\}+\begin{bmatrix} \diagdown & & \\ & \overline{K} & \\ & & \diagdown \end{bmatrix}\{p\}=[U]^T\{F\}$$

with diagonal matrices of modal mass, modal stiffness and, under certain conditions, modal damping. The mode shapes will uncouple the mass and stiffness matrices and for certain specific types of damping, these mode shapes will also uncouple the damping matrix. In order to understand some of these conditions, a simple example will be shown.

For the example here, the matrices will be defined with

$$[M]=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad ; \quad [K]=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[C_0]=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad [C_P]=\begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.4 \end{bmatrix} \quad ; \quad [C_N]=\begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$$

First, the undamped case is considered. The mass, [M], and stiffness, [K], will be used with the $[C_0]$ matrix. The eigensolution of this set of matrices will yield frequencies, residues and shapes as:

$$\lambda_1 = 0+0.3737j; \quad a_1=\begin{Bmatrix} 0+0.1230j \\ 0+0.2116j \end{Bmatrix}; \quad u_1=\begin{Bmatrix} 0.2459 \\ 0.4232 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1.72 \end{Bmatrix}$$

$$\lambda_2 = 0+1.0926j; \quad a_2=\begin{Bmatrix} 0+0.1868j \\ 0-0.0724j \end{Bmatrix}; \quad u_2=\begin{Bmatrix} 0.3735 \\ -0.1447 \end{Bmatrix}=\begin{Bmatrix} 1 \\ -0.387 \end{Bmatrix}$$

Notice that the mode shape is a sign valued (+ or -) real number. The first mode has both DOFs with the same sign indicating that both of these DOFs are in phase with each other differing only in magnitude. The second mode has both DOF with differing signs indicating that both DOFs are out of phase with each other and have differing magnitudes.

Now let's consider the second case with damping which is proportional to either the mass and/or stiffness of the system. The damping here is $[C_P]$ to be used with the [M] and [K]. The eigensolution of this set of matrices will yield frequencies, residues and shapes as:

$$\lambda_1 = -0.0579+0.3693j \quad a_1=\begin{Bmatrix} 0+0.1244j \\ 0+0.2141j \end{Bmatrix} \quad u_1=\begin{Bmatrix} 0.2488 \\ 0.4282 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1.72 \end{Bmatrix}$$

$$\lambda_2 = -0.1097+1.0871j \quad a_2=\begin{Bmatrix} 0+0.1877j \\ 0-0.0727j \end{Bmatrix} \quad u_2=\begin{Bmatrix} 0.3754 \\ -0.1455 \end{Bmatrix}=\begin{Bmatrix} 1 \\ -0.387 \end{Bmatrix}$$

Notice that the eigensolution yields the same mode shapes as the undamped case. This is due to the fact that the damping is proportional to the mass and /or stiffness of the system. This results in modes that are referred to as "real normal modes". So it is clear that the mode shapes for the undamped and proportionally damped cases are exactly the same.

Now let's consider the third case with damping which is not proportional to either the mass and/or stiffness of the system. The damping here is $[C_N]$ to be used with the [M] and [K]. The eigensolution of this set of matrices will yield frequencies, residues and shapes as:

$$\lambda_1 = -0.0162 + 0.3736j \quad a_1 = \begin{Bmatrix} -0.0071 + 0.1288j \\ -0.0048 + 0.2116j \end{Bmatrix} \quad u_1 = \begin{Bmatrix} 0.2456 + 0.0143j \\ 0.4232 + 0.0095j \end{Bmatrix}$$

$$\lambda_2 = -0.1005 + 1.0872j \quad a_2 = \begin{Bmatrix} -0.0071 + 0.1885j \\ -0.0048 - 0.0726j \end{Bmatrix} \quad u_2 = \begin{Bmatrix} -0.3771 + 0.0142j \\ 0.1451 + 0.0096j \end{Bmatrix}$$

Now for this case, the mode shapes are seen to be different than the previous cases. First of all, the mode shapes are complex valued. Upon closer inspection of these shapes, it can be seen that the relative phasing between each DOF for each of the modes is NOT either totally in-phase or out-of-phase. This results in modes which are described as "complex modes". This is very different from the two previous cases. This will typically occur when the damping for the system is not related to the mass and/or stiffness of the system and is referred to as non-proportional damping. In order to perform the eigen solution, a slightly different form is used where the equations are cast in state space in order to perform the solution.

Basically all the equations get more complicated when considering complex modes. Some simple statements between a real normal mode and a complex mode can be summarized as follows:

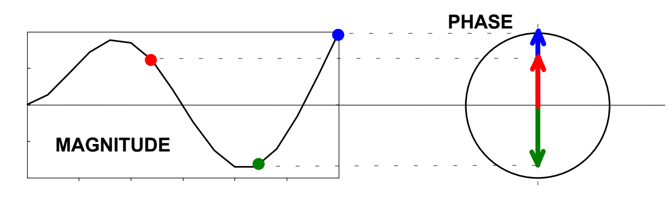

Figure 1a - Proportional (Real Normal) Mode Schematic

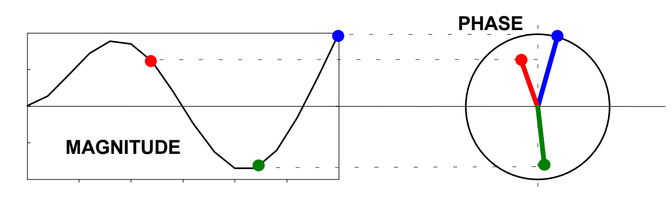

Figure 1b - Non-Proportional (Complex) Mode Schematic

REAL NORMAL MODE

Some characteristics of a real normal mode are:

1. The mode shape is described by a standing wave which has the presence of a fixed stationary node point
2. All points pass through their maxima and minima at the same instant in time
3. All points pass through zero at the same instant in time
4. The mode shape can be described as a sign valued, real number
5. All points are either totally in-phase or out-of-phase with any other point on the structure
6. The mode shapes from the undamped case are the same as the proportionally damped case. These shapes uncouple the [M],[C], and [K]

COMPLEX MODE

Some characteristics of a complex mode are:

1. The mode shape is described by a traveling wave and appears to have a moving node point on the structure
2. All points do not pass through their maxima at the same instant in time - points appear to lag behind other points
3. All points do not pass through zero at the same instant in time
4. The mode shape can not be described by real valued numbers - the shapes are complex valued
5. The different DOFs will have some general phase relationship that will not necessarily be in-phase or 180 degrees out-of-phase with other DOF
6. The mode shapes from the undamped case will not uncouple the damping matrix

In order to further visualize some of these statements. A simple mode shape is plotted for a real normal mode and a complex mode for one of the modes of a cantilever beam. In the real normal mode (Figure 1a), the relative phasing between the DOF is either totally in phase (as in the case of the blue and red DOF) or totally out of phase by 180 degrees (as in the case of the green DOF relative to the blue and red DOF). A complex mode does not have this simple phasing relationship and the mode shape must be described by both amplitude and phase, or real and imaginary components (Figure 1b). The plots in Figure 1 are intended to visualize this relationship of the phase.

Now it is very important to point out that phasing can be seen in FRF measurements all the time. Sometimes this may be an indication of complex mode behavior, but be careful to jump conclusions. The data acquisition, instrumentation, signal processing, FFT, and modal parameter estimation are all stages that can distort a measurement and force a mode shape to "appear" as if it is complex.

While there is a lot more to it all, I hope this simple explanation helps to put everything in better perspective. Think about it and if you have any more questions about modal analysis, just ask me.