Presentation Topics

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Modal Analysis & Controls Laboratory

University of Massachusetts Lowell

IMAC 19
Young Engineer Program

TUTORIAL:

Basics of Modal Analysis

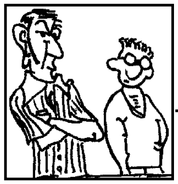
Intent of Young Engineer Program

The intent of the Young Engineer Program is to expose the new or young engineer to some of the basic concepts and ideas concerning analytical and experimental modal analysis.

It is NOT intended to be a detailed treatment of this material.

Rather it is intended to prepare one for some of the in-depth papers presented at IMAC so that the novice has some appreciation of the detailed material presented in these papers.

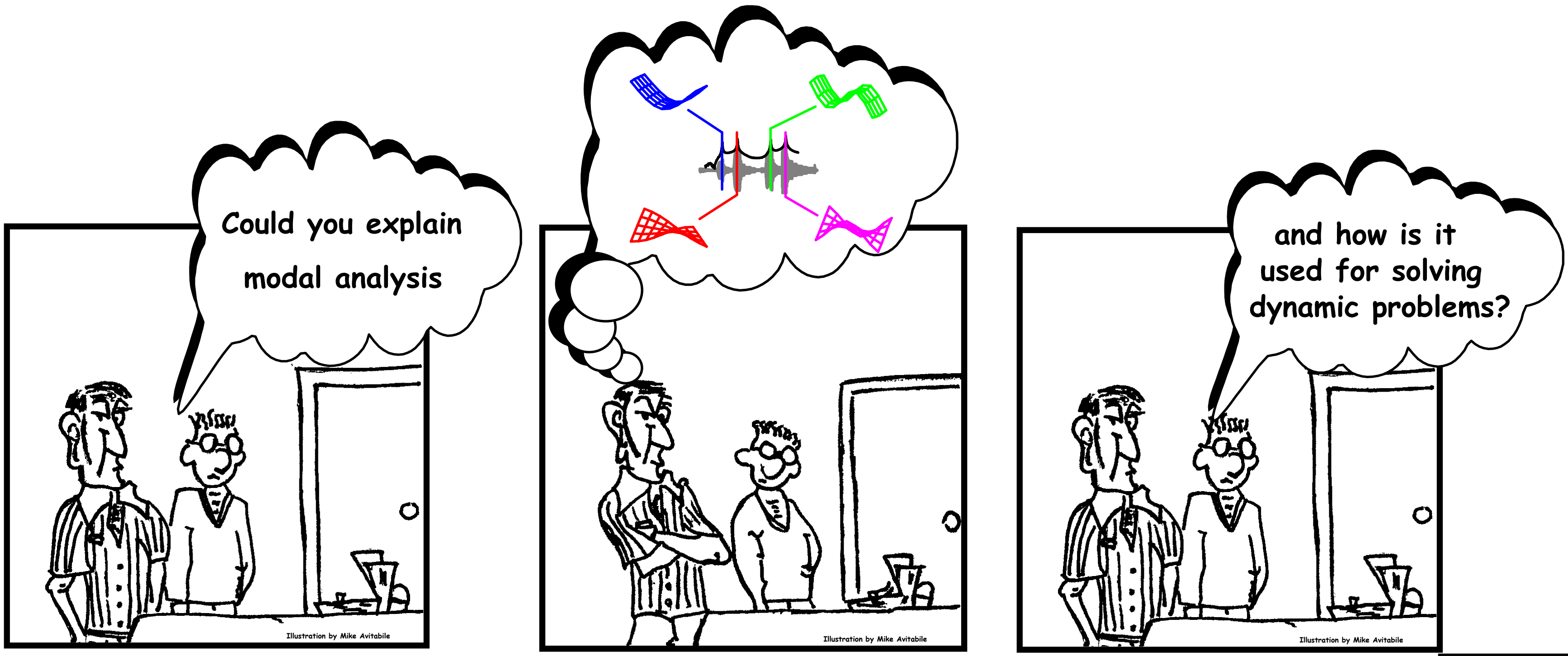
This presentation is intended to identify the basic methodology and techniques currently employed in this field and to expose one to the typical modal jargon used in the field.



A Simple Non-Mathematical Presentation

Dr. Peter Avitabile

Mechanical Engineering - UMASS Lowell



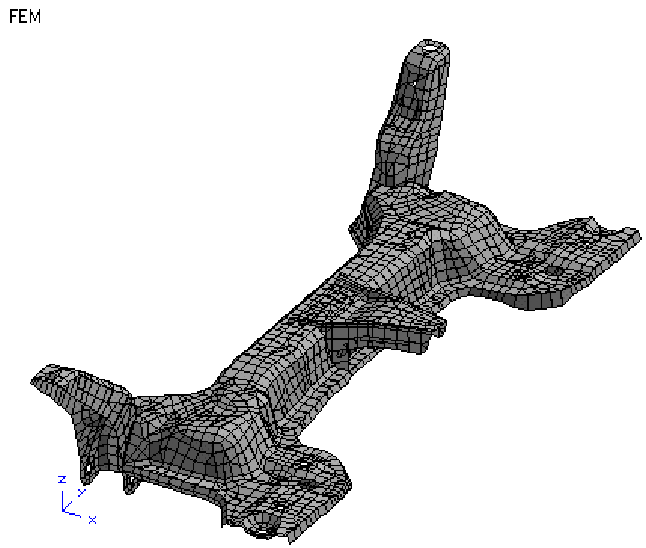
Analytical Modal Analysis

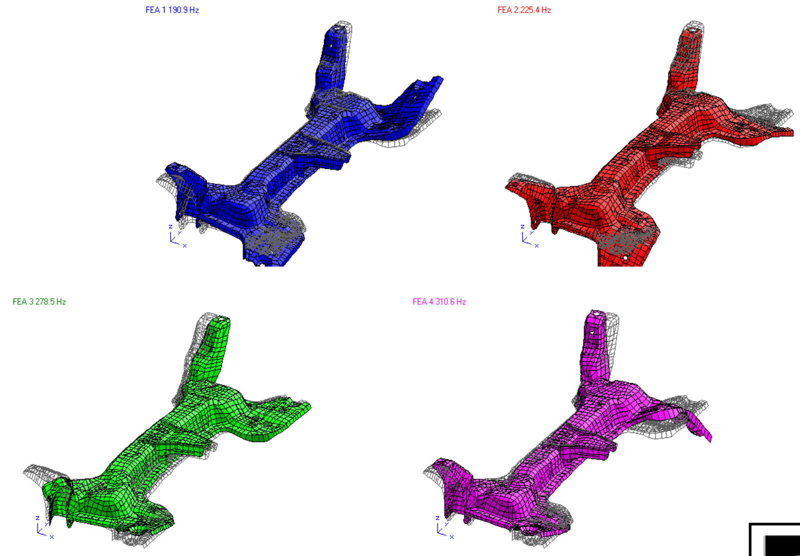
Equation of motion

$$[M_n]\{\ddot{x}_n\}+[C_n]\{\dot{x}_n\}+[K_n]\{x_n\}=\{F_n(t)\}$$

Eigensolution

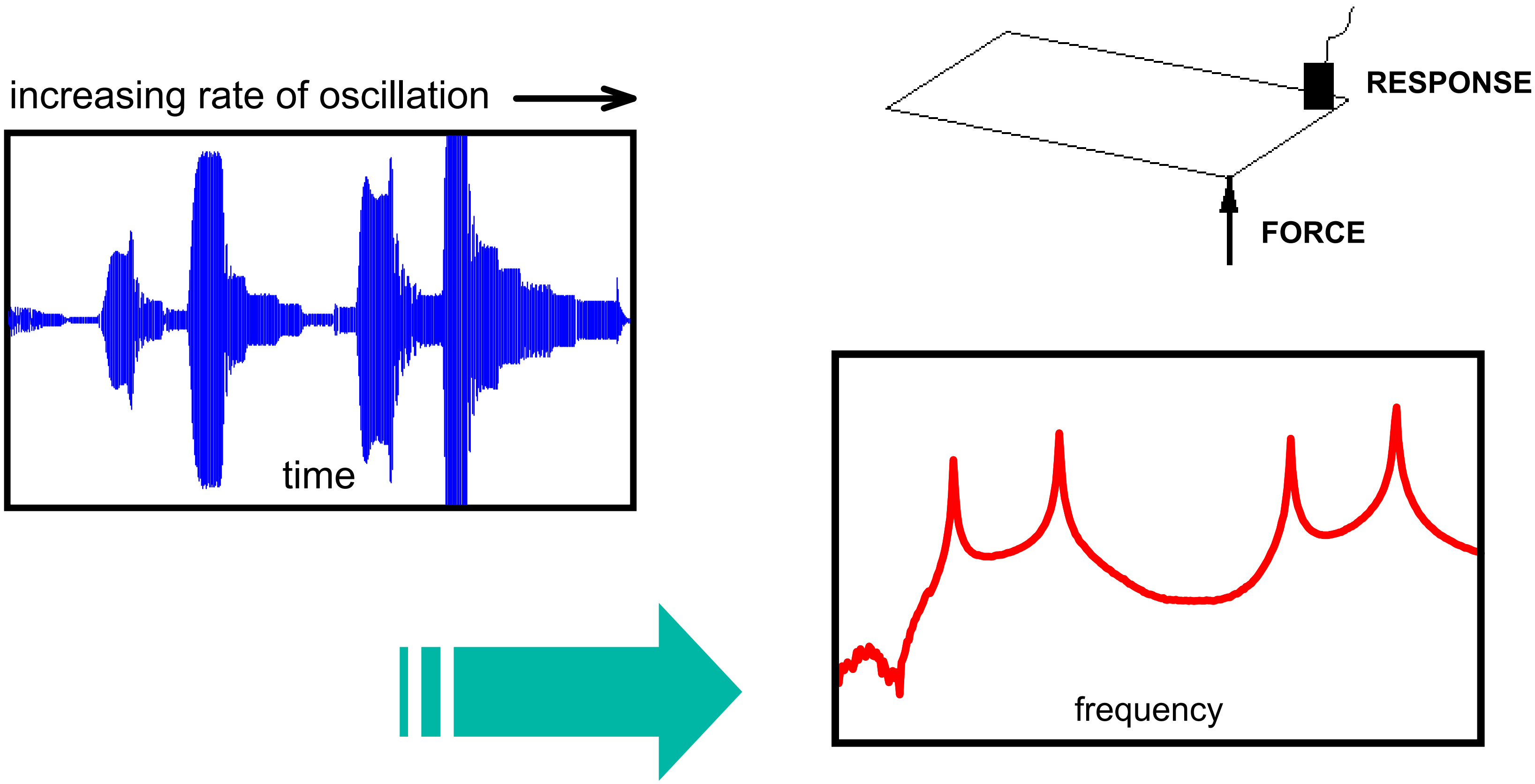
$$[[K_n]-\lambda[M_n]]\{x_n\}=\{0\}$$





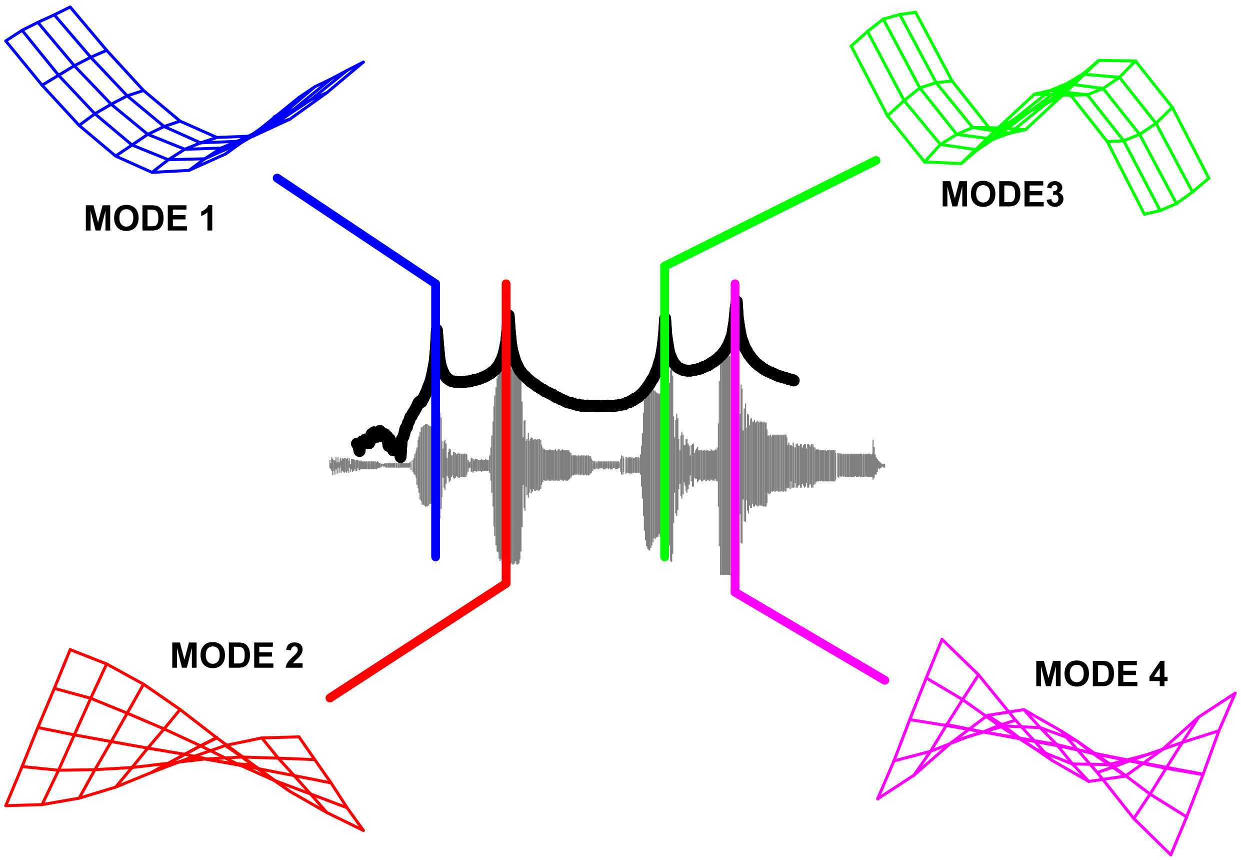
Could you explain modal analysis for me ?

Simple time-frequency response relationship

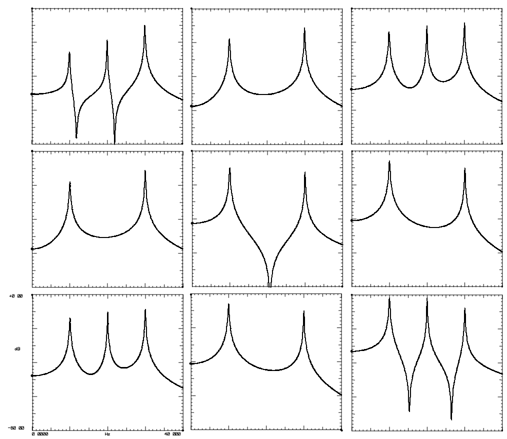


Could you explain modal analysis for me ?

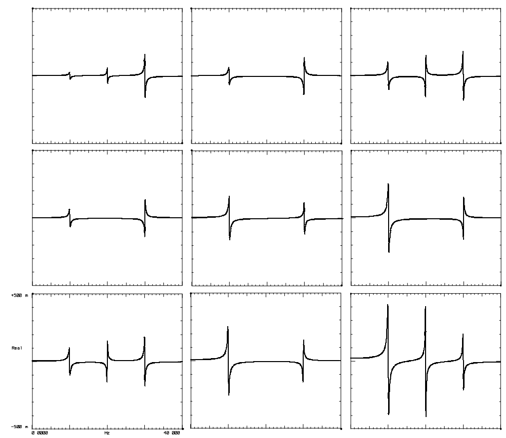
Sine Dwell to Obtain Mode Shape Characteristics



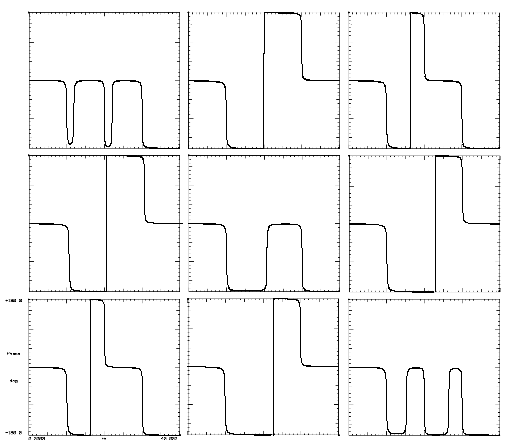
Just what are the measurements called FRFs ?



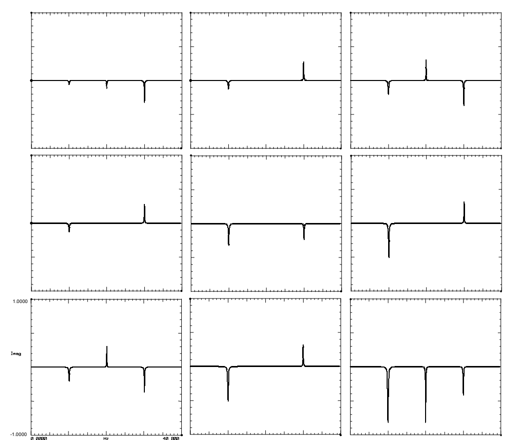
Magnitude



Real

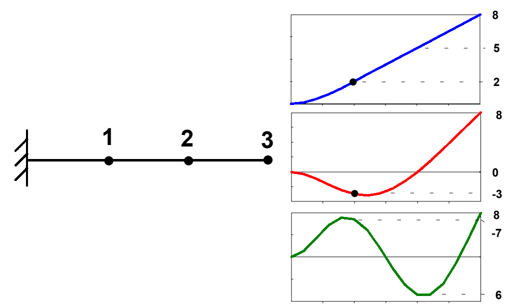


Phase



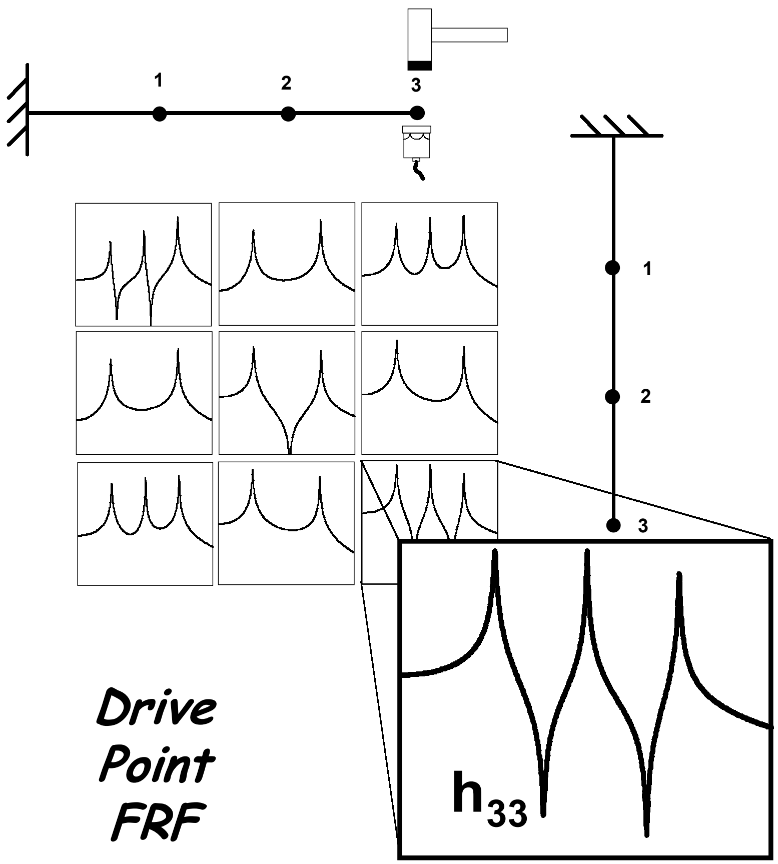
Imaginary

A simple input-output problem

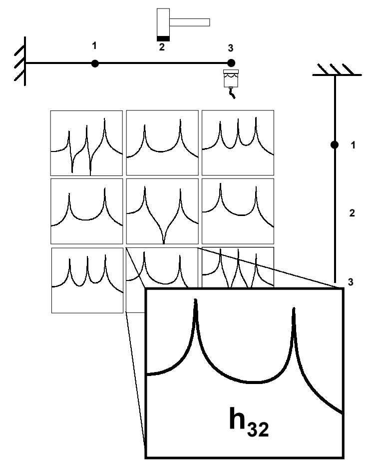


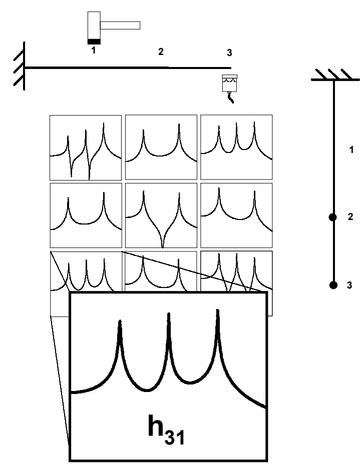
Just what are the measurements called FRFs ?

Response at point 3
due to an input at point 3



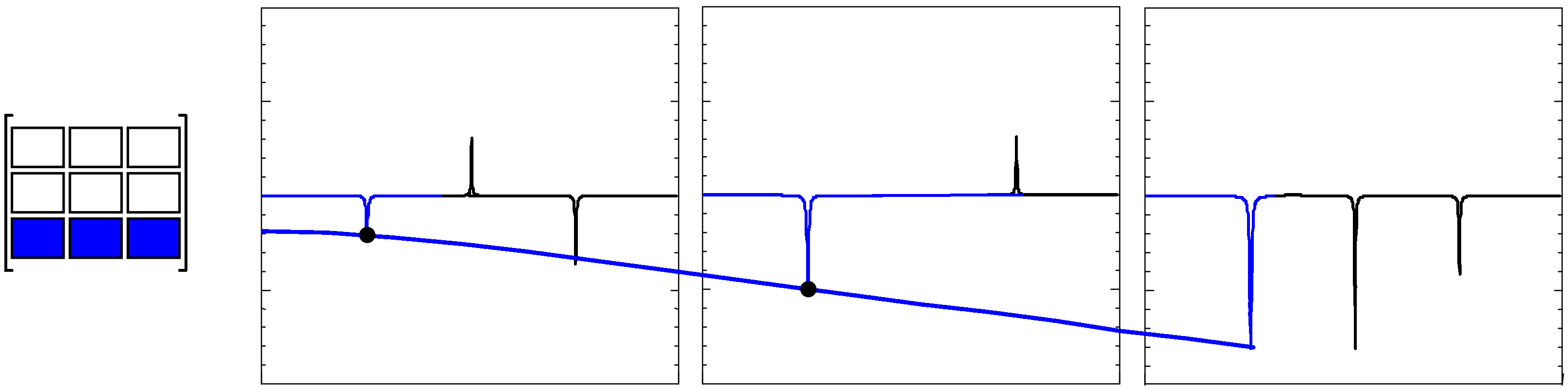
Drive Point FRF





Why is only one row/column of FRFs needed ?

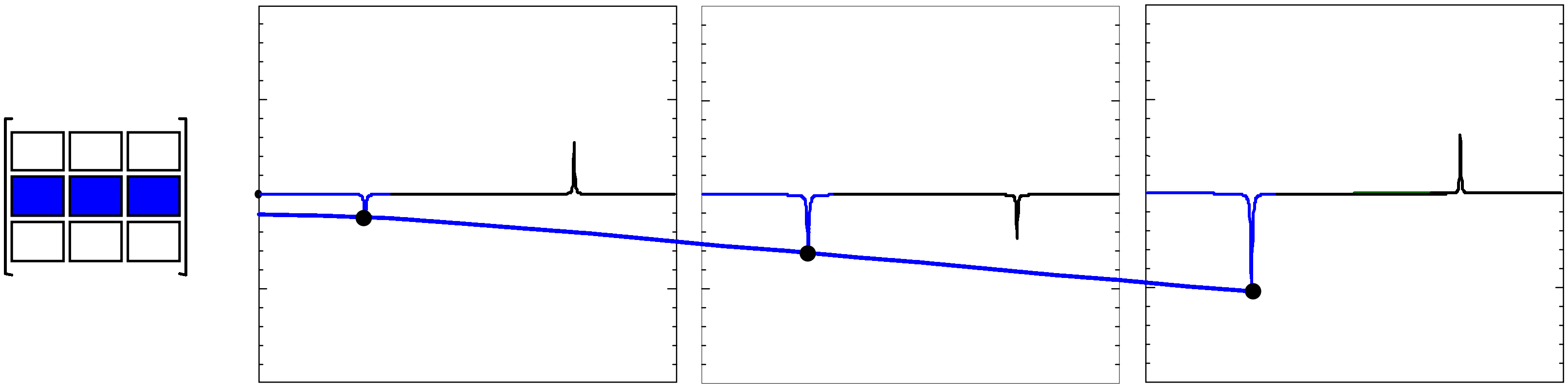
The third row of the FRF matrix - mode 1



The peak amplitude of the imaginary part of the FRF is a simple method to determine the mode shape of the system

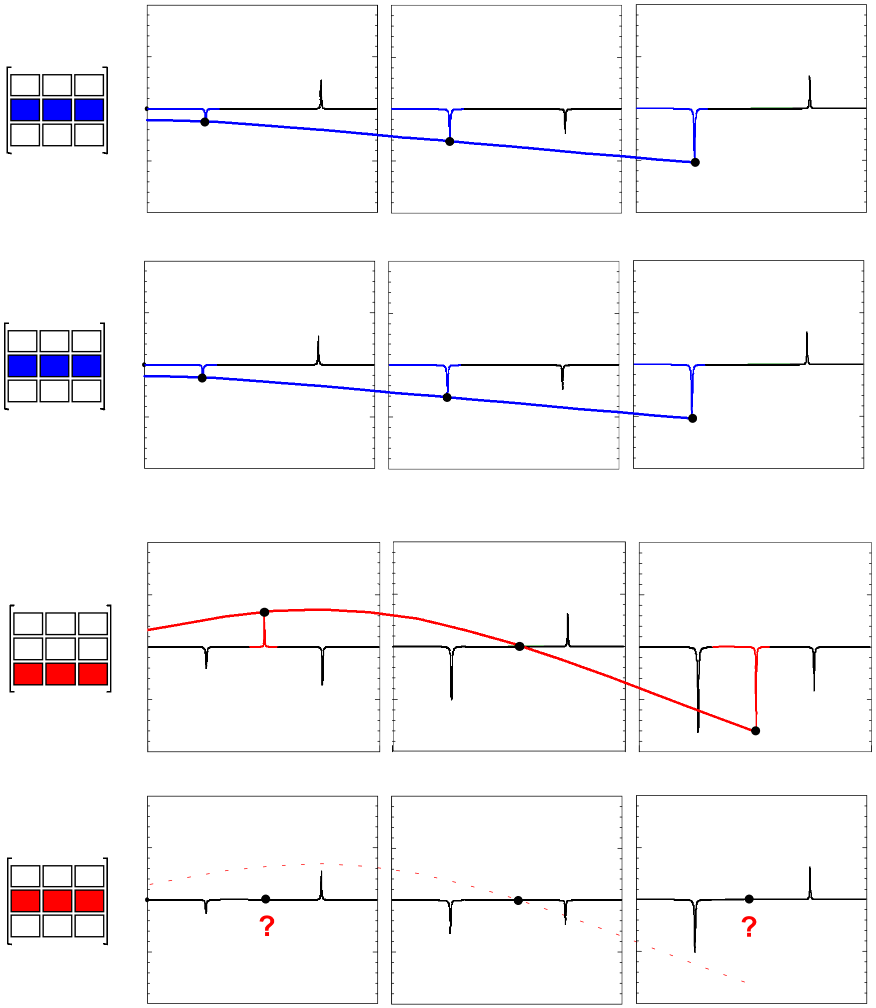
Why is only one row/column of FRFs needed ?

The second row of the FRF matrix is similar



The peak amplitude of the imaginary part of the FRF is a simple method to determine the mode shape of the system

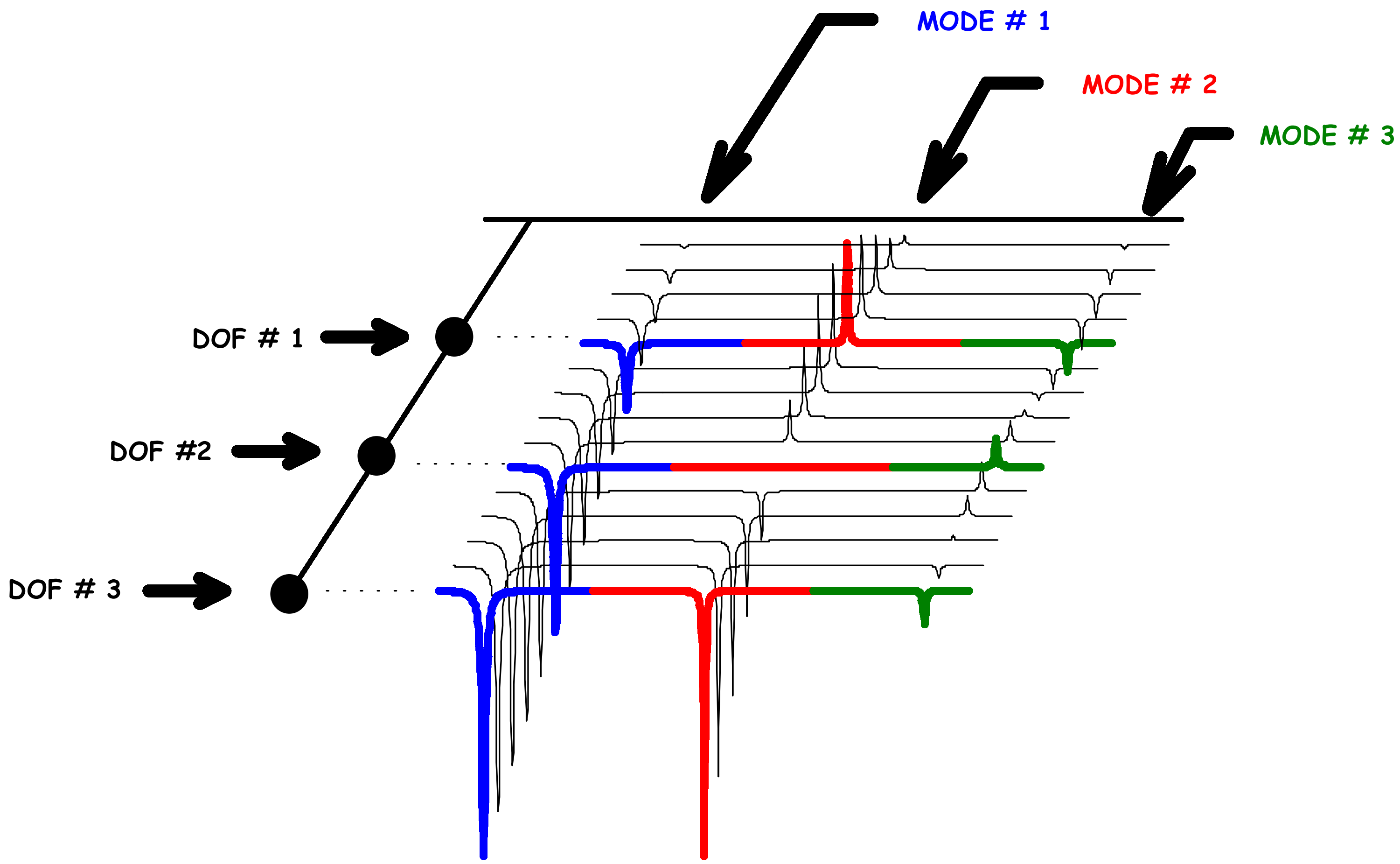
Why is only one row/column of FRFs needed ?



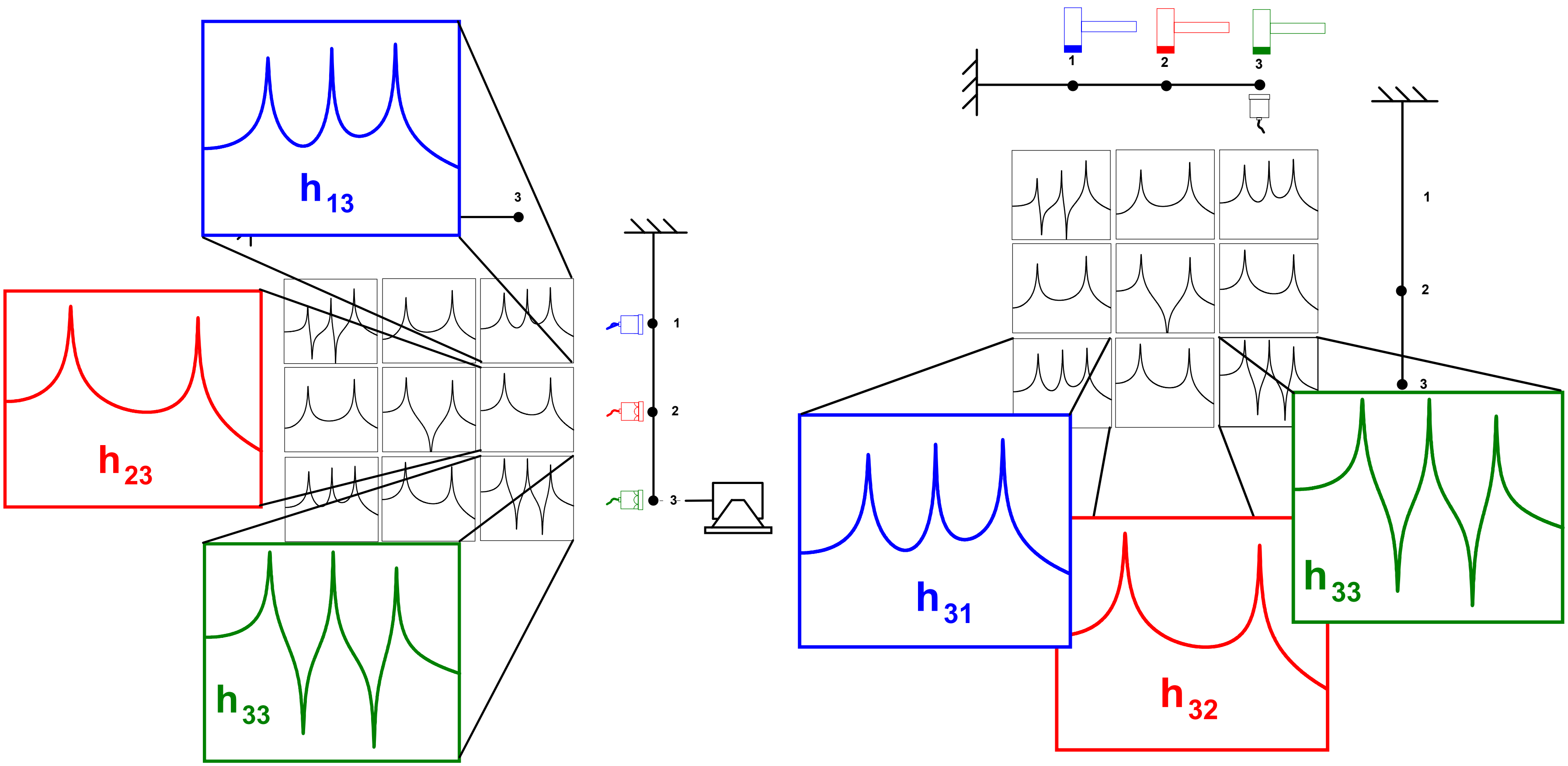
Any row or column can be used to extract mode shapes

- as long as it is not the node of a mode !

More measurements better defines the shape

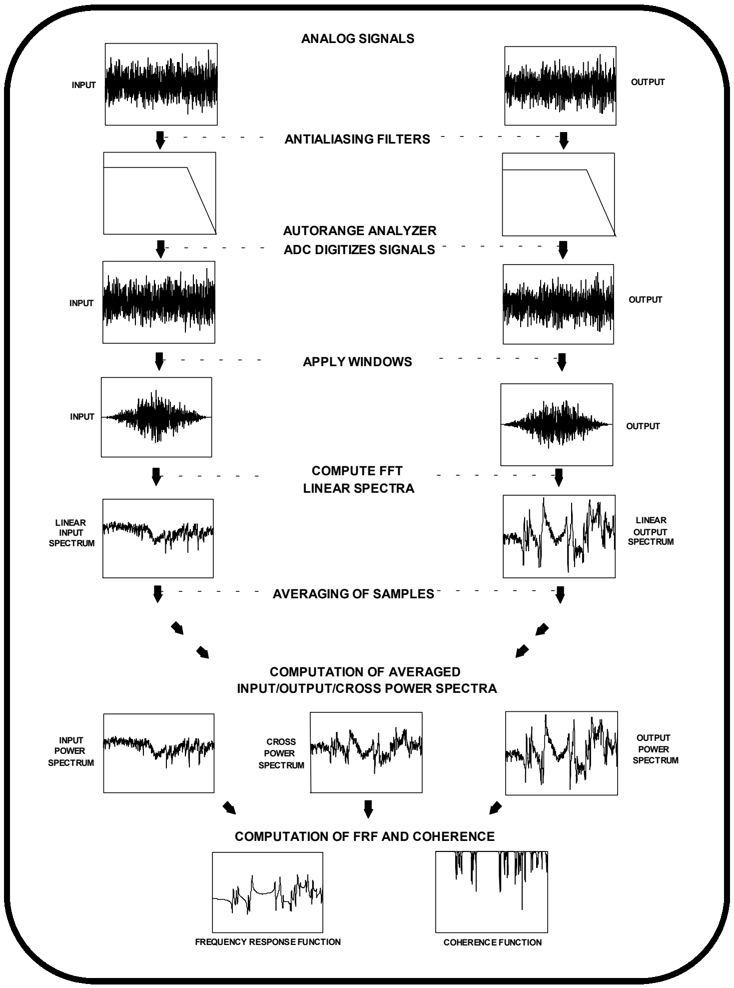


What's the difference between shaker and impact ?



Theoretically - - - NOTHING ! ! !

What measurements do I actually make ?



Actual time signals

Analog anti-alias filter

Digitized time signals

Windowed time signals

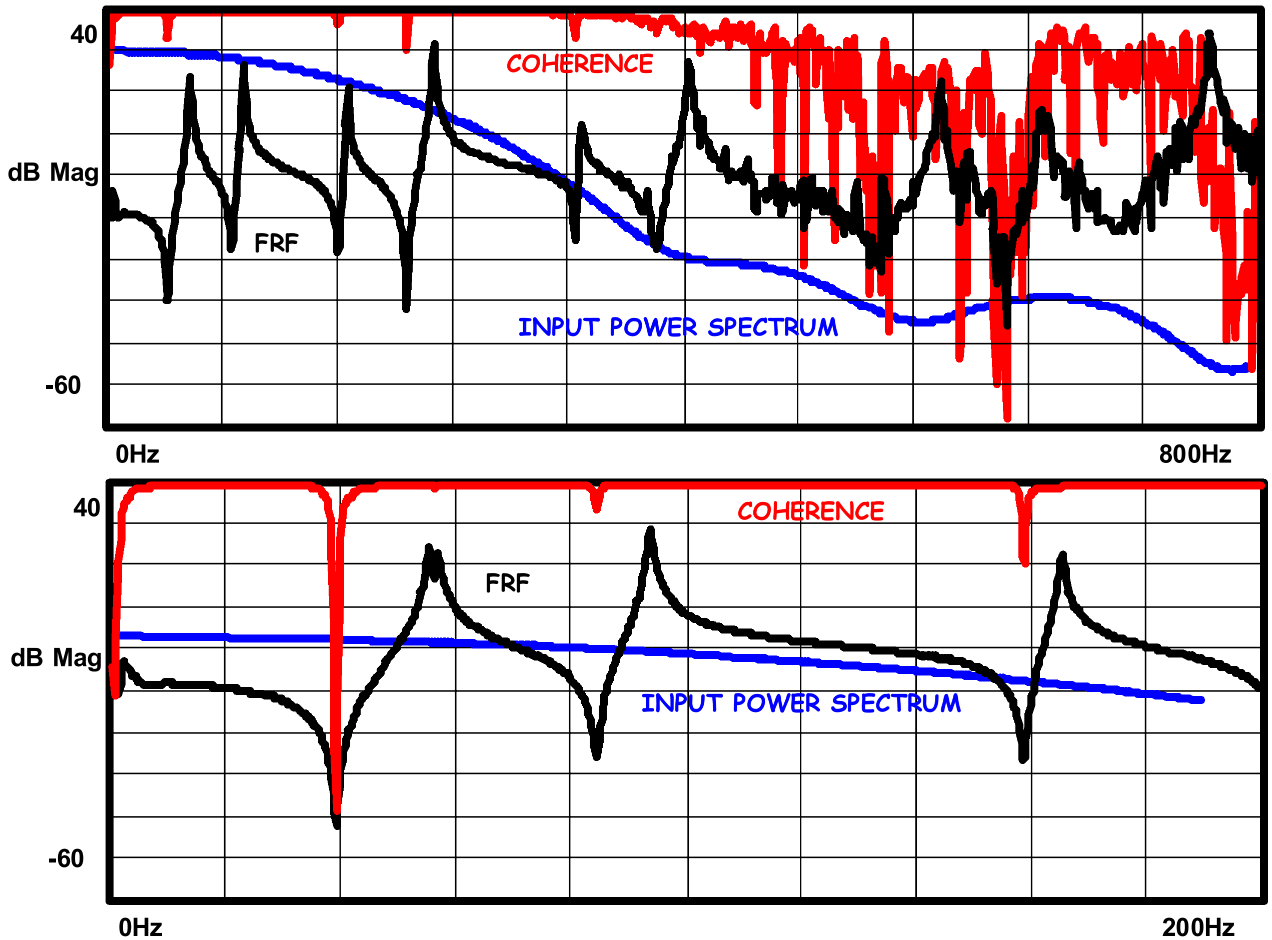
Compute FFT of signal

Average auto/cross spectra

Compute FRF and Coherence

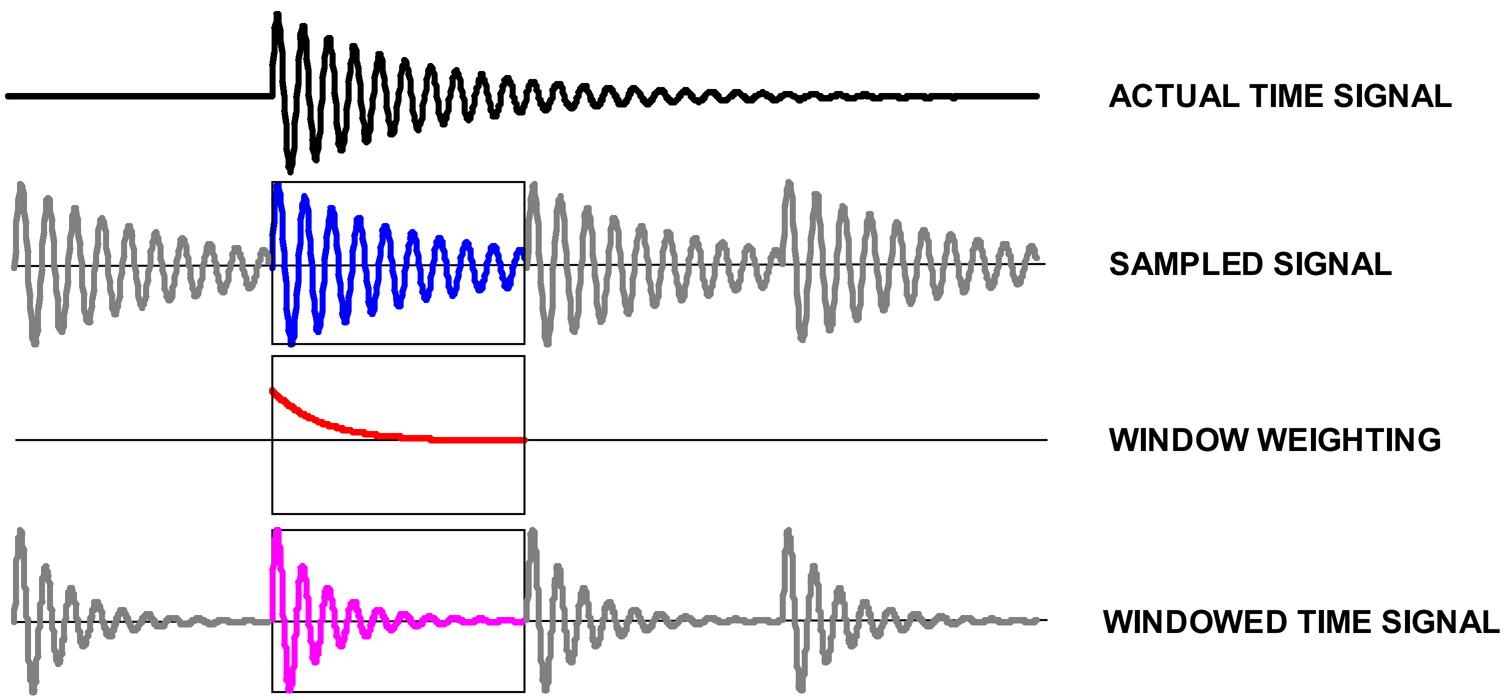
What's most important in impact testing ?

Hammers and Tips

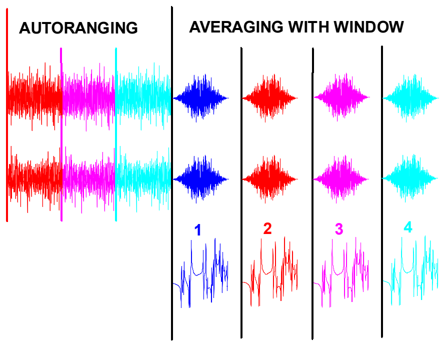


What's most important in impact testing ?

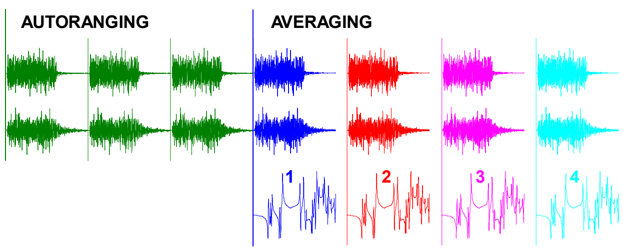
Leakage and Windows



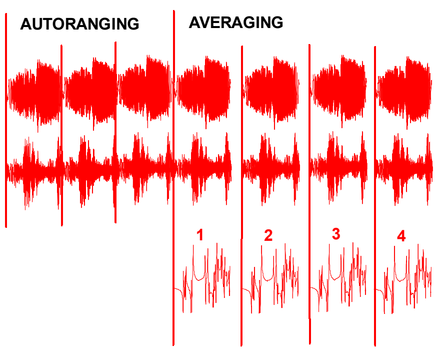
What's most important in shaker testing ?



Random with Hanning



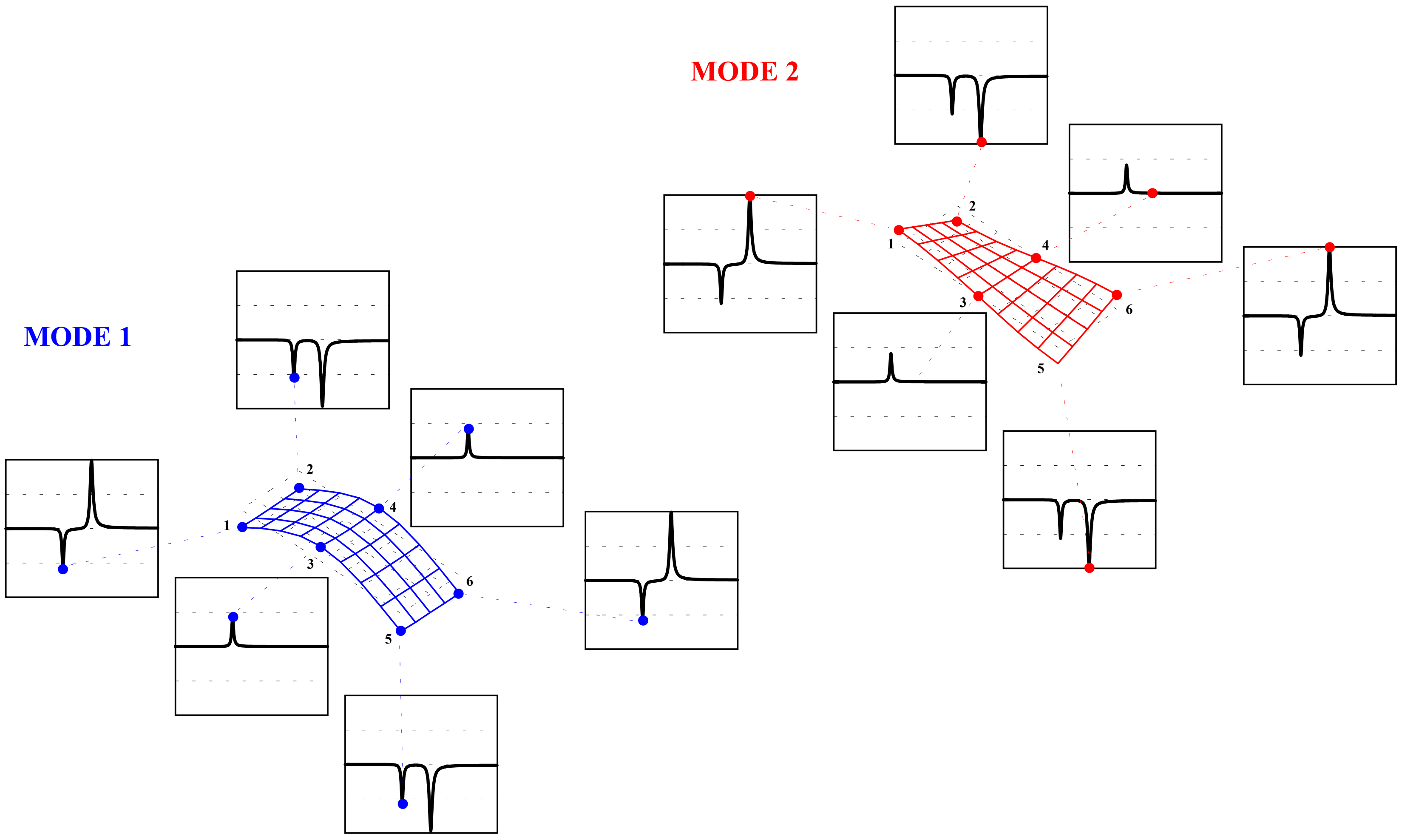
Burst Random



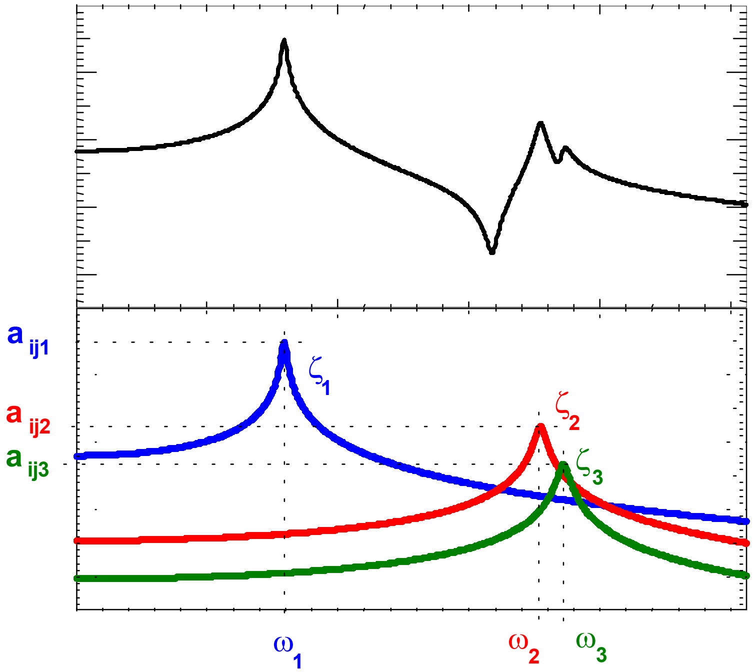
Sine Chirp

Different excitation techniques are available for obtaining good measurements

How do I get mode shapes from the FRFs ?



How do I get mode shapes from the FRFs ?



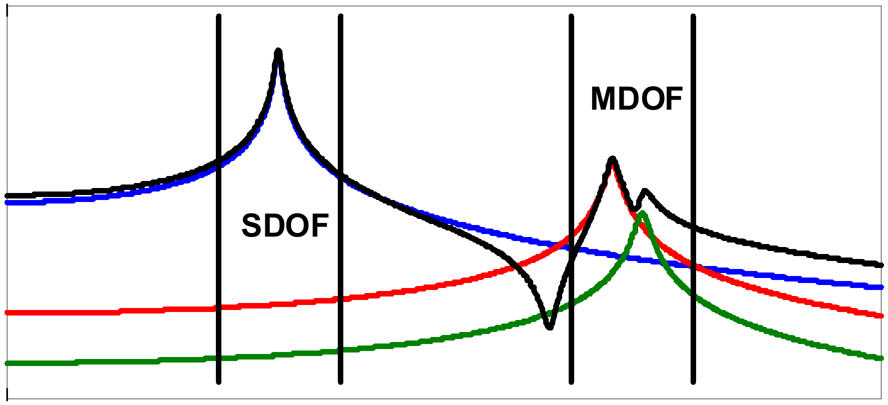
The FRF is made up from each individual mode contribution which is determined from the

frequency,

damping,

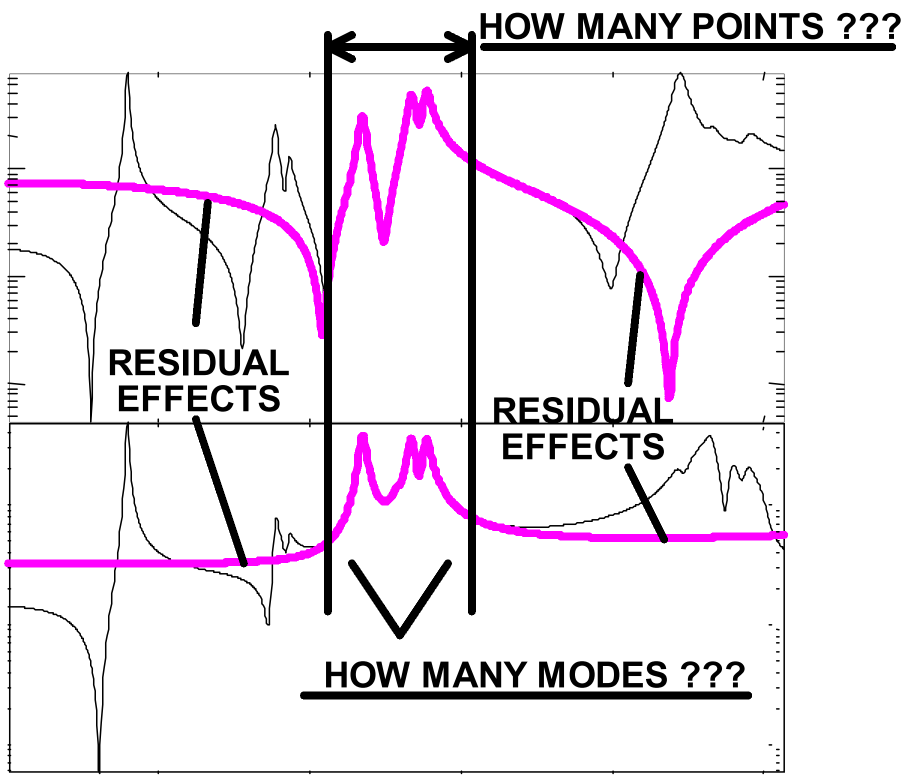
residue

How do I get mode shapes from the FRFs ?



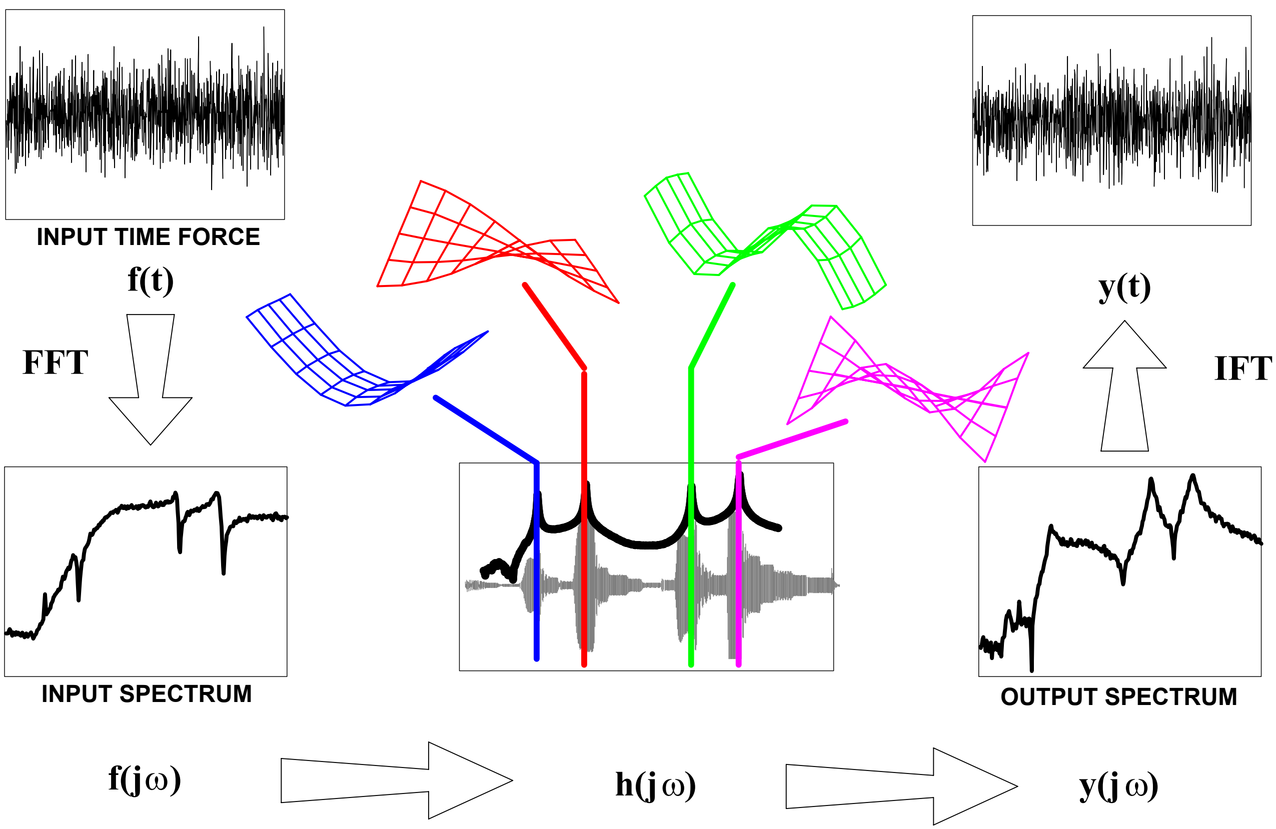
The task for the modal test engineer is to determine the parameters that make up the pieces of the frequency response function

How do I get mode shapes from the FRFs ?



Mathematical routines help to determine the basic parameters that make up the FRF

Why and How Do Structures Vibrate?

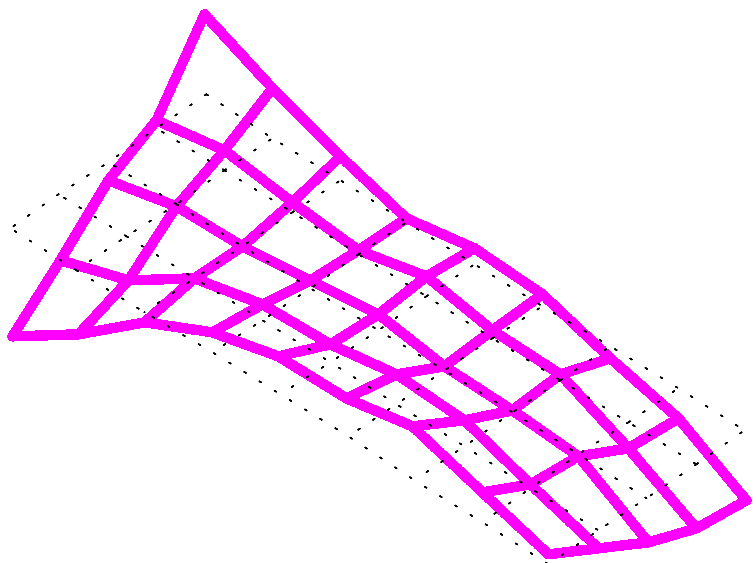


What is operating data ?

If I make measurements on a structure at an operating frequency, sometimes I get some deformation shapes that look pretty funky .

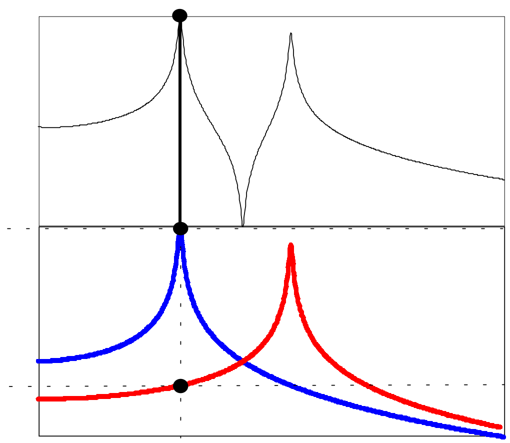
Maybe they are just noise?

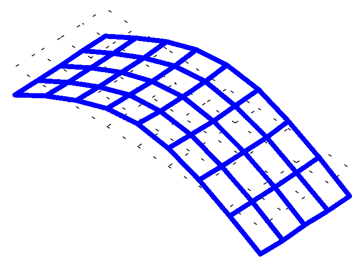
Is that possible ???

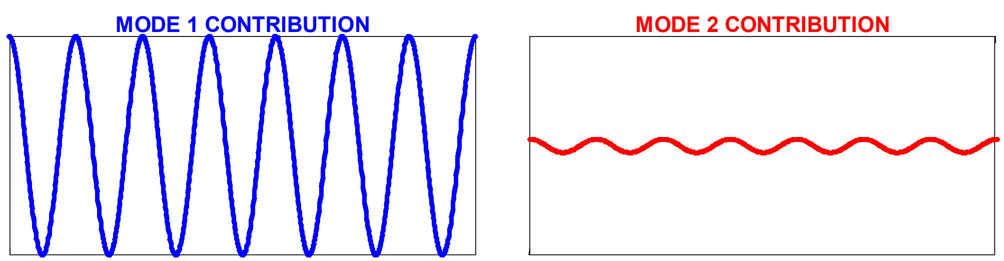


What is operating data ?

But if I make a measurement at an operating frequency and its close to a mode, I can easily see what appears to be one of the modes

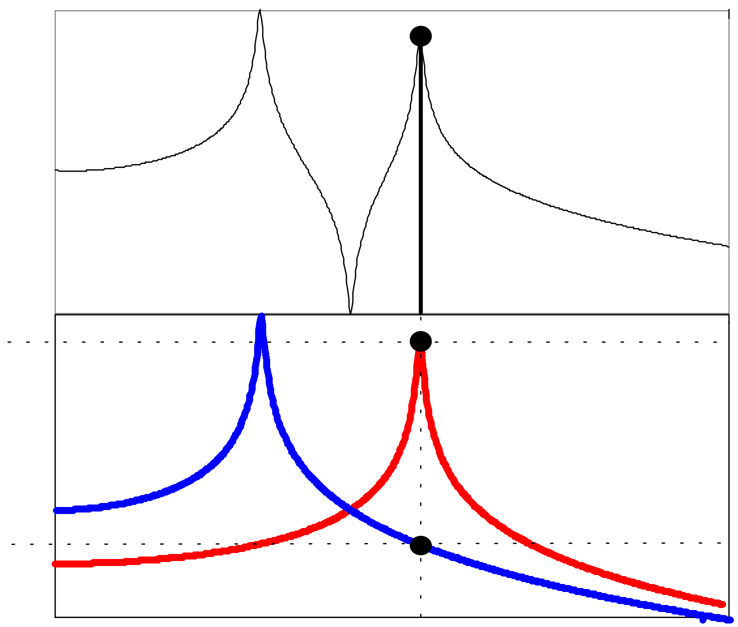


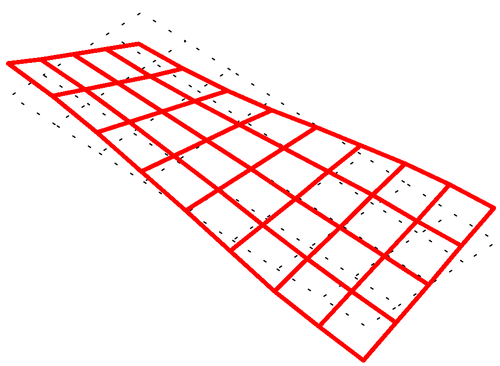




What is operating data ?

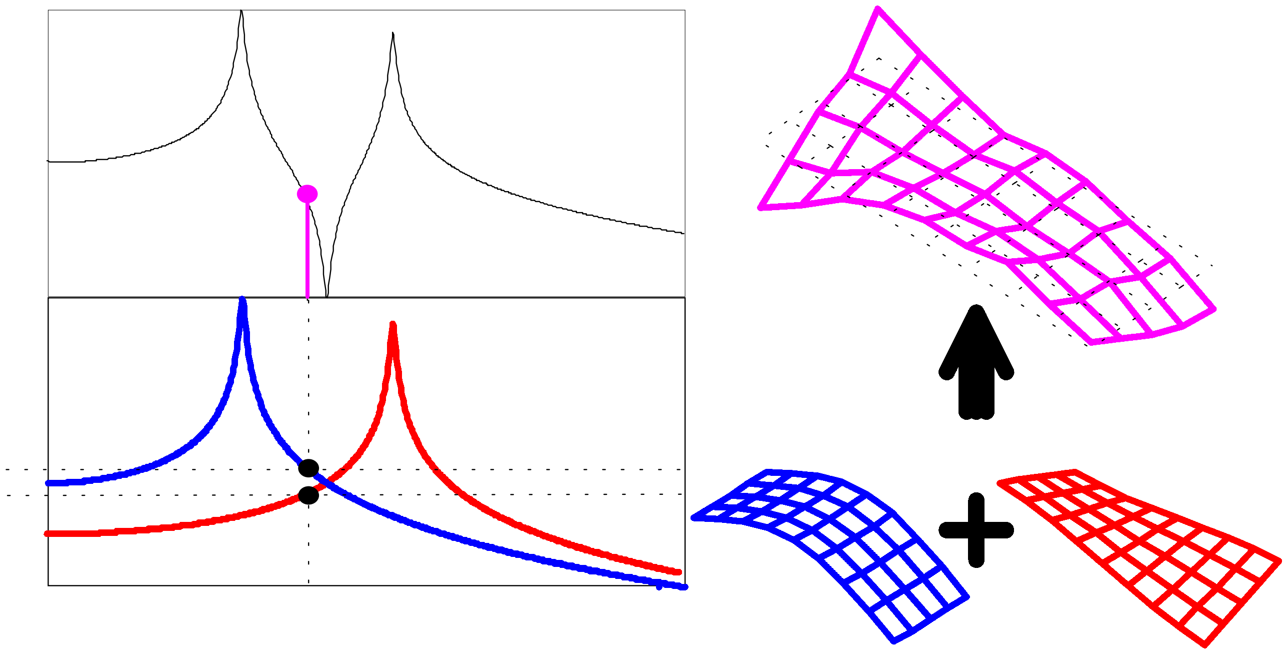
And if I make a measurement at an operating frequency and its close to another mode, I can easily see what appears to be one of the modes





What is operating data ?

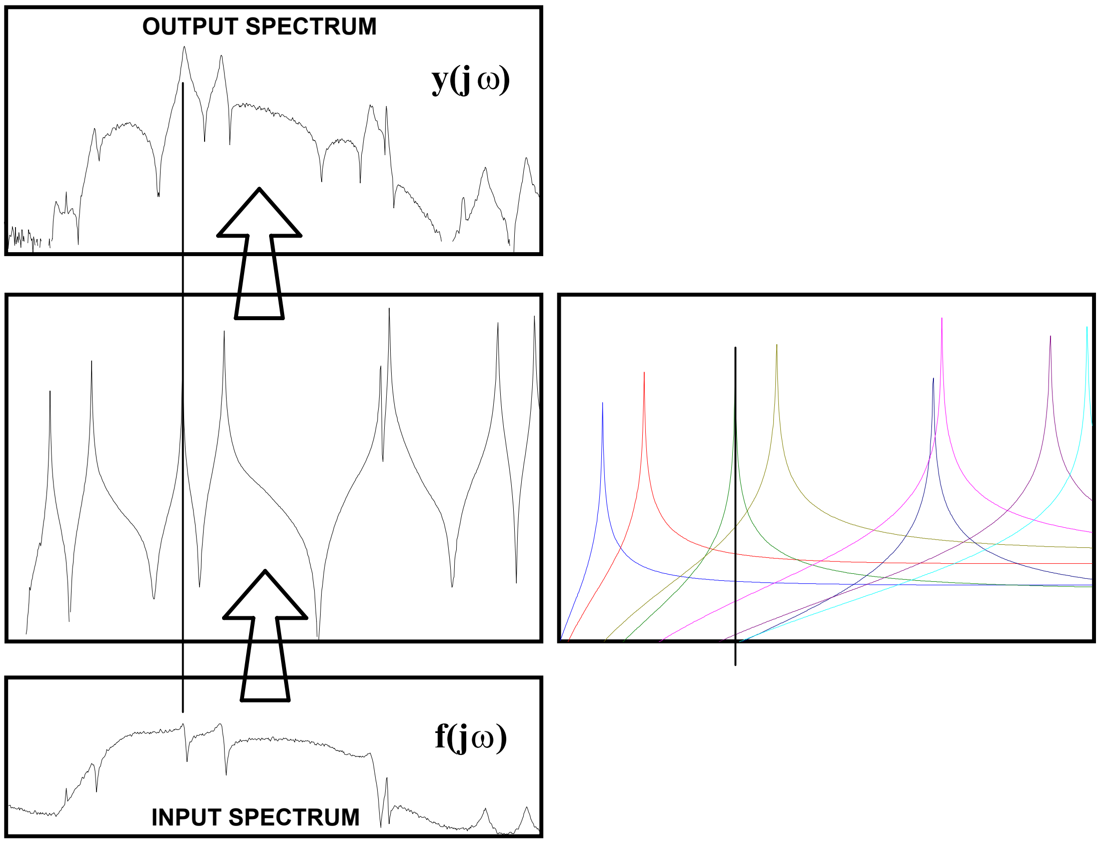
I think I just answered my own question !!!



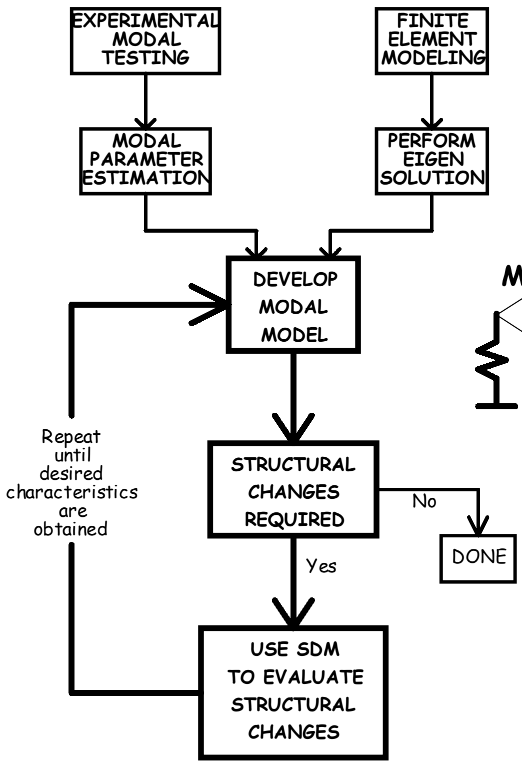
I think I'm starting to understand this now !!!

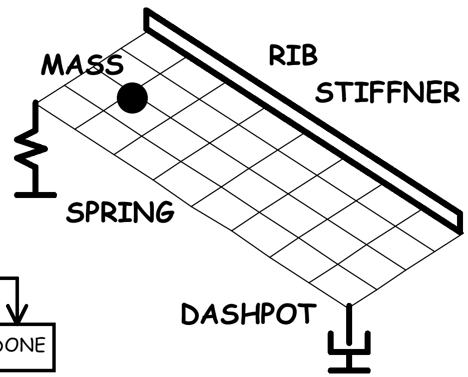
What is operating data ?

The modes of the structure act like filters which amplify and attenuate input excitations on a frequency basis



So what good is modal analysis ?



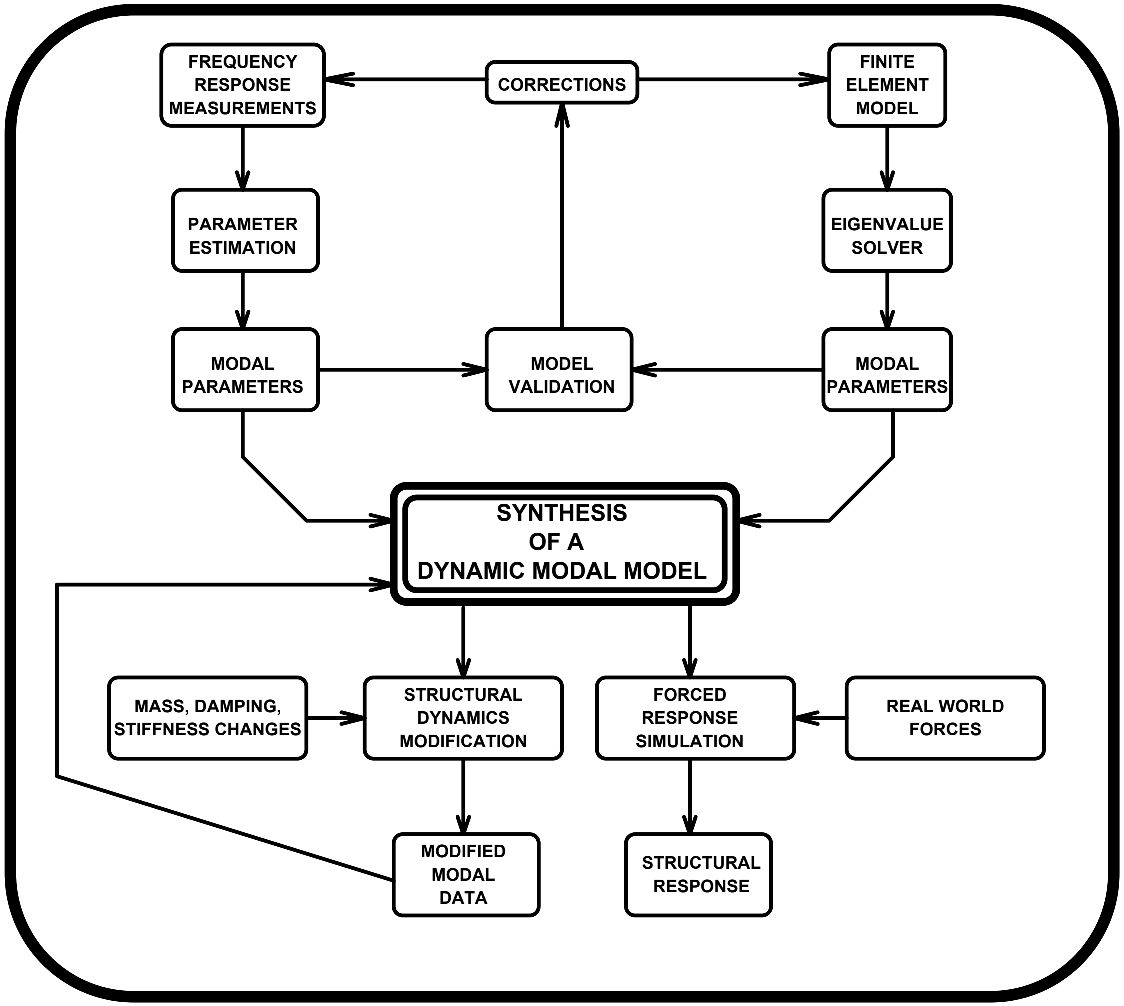


STRUCTURAL
DYNAMIC
MODIFICATIONS

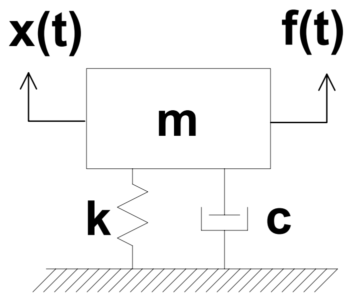
The dynamic model can be used for studies to determine the effect of structural changes of the mass, damping and stiffness

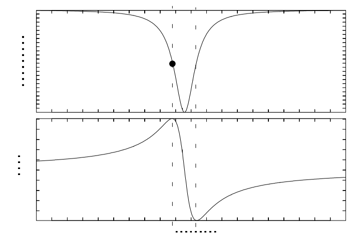
So what good is modal analysis ?

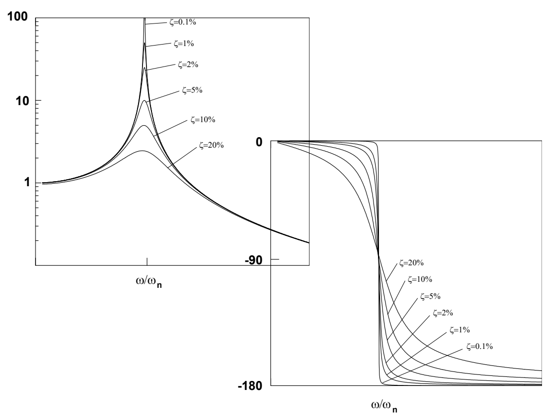
Simulation, Prediction, Correlation, … to name a few

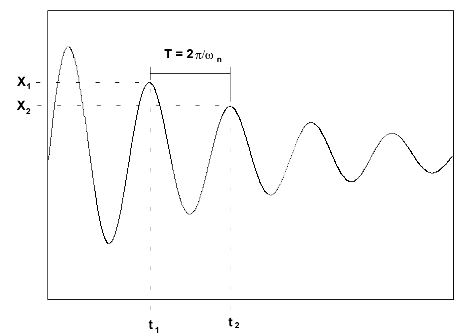


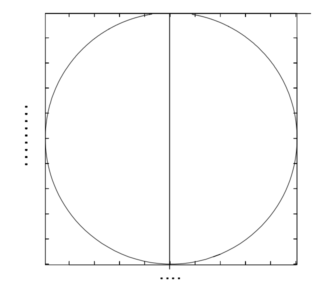
Single Degree of Freedom Overview









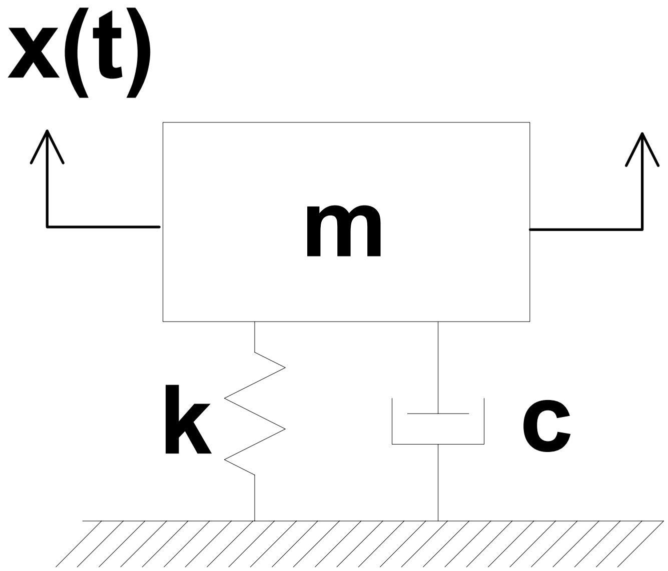


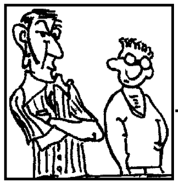
$$h(s) = \frac{1}{ms^2 + cs + k}$$

SDOF Definitions

Assumptions

- lumped mass
- stiffness proportional to displacement
- damping proportional to velocity
- linear time invariant
- 2nd order differential equations





SDOF Equations

Equation of Motion

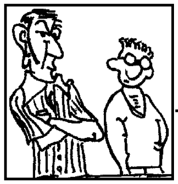
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \qquad \textit{or} \qquad m\ddot{x} + c\dot{x} + kx = f(t)$$

Characteristic Equation

$$ms^2 + cs + k = 0$$

Roots or poles of the characteristic equation

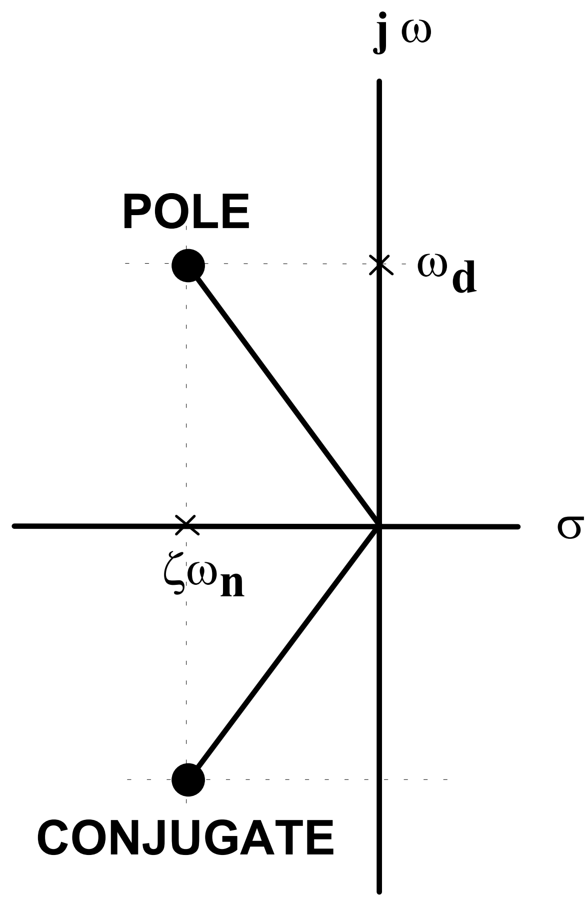
$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \frac{k}{m}}$$



SDOF Definitions

Poles expressed as

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\sigma \pm j\omega_d$$



Damping Factor $\sigma = \zeta\omega_n$

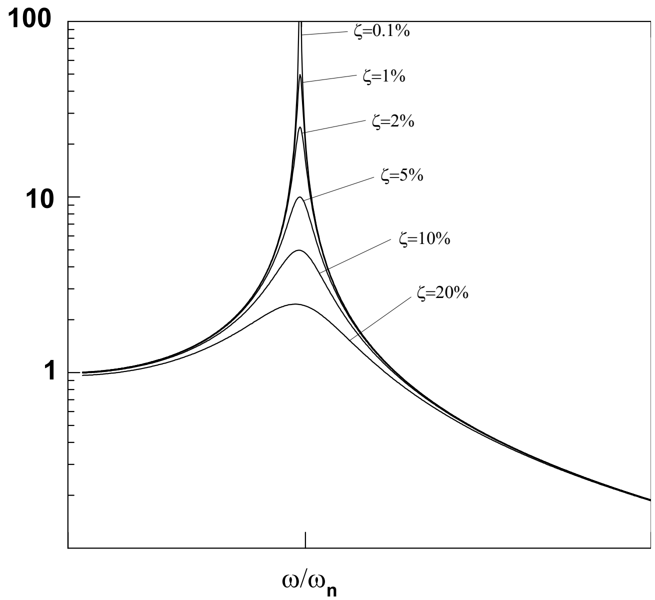
Natural Frequency $\omega_n = \sqrt{k/m}$

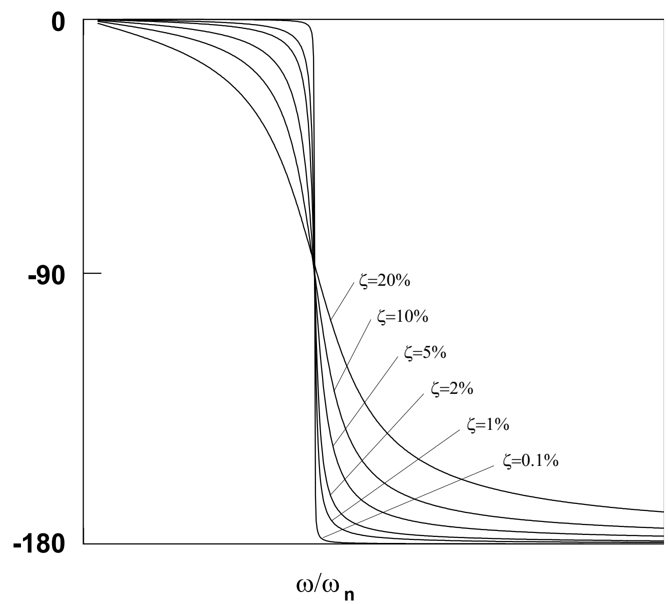
% Critical Damping $\zeta = c/c_c$

Critical Damping $c_c = 2m\omega_n$

Damped Natural Frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

SDOF - Harmonic Excitation

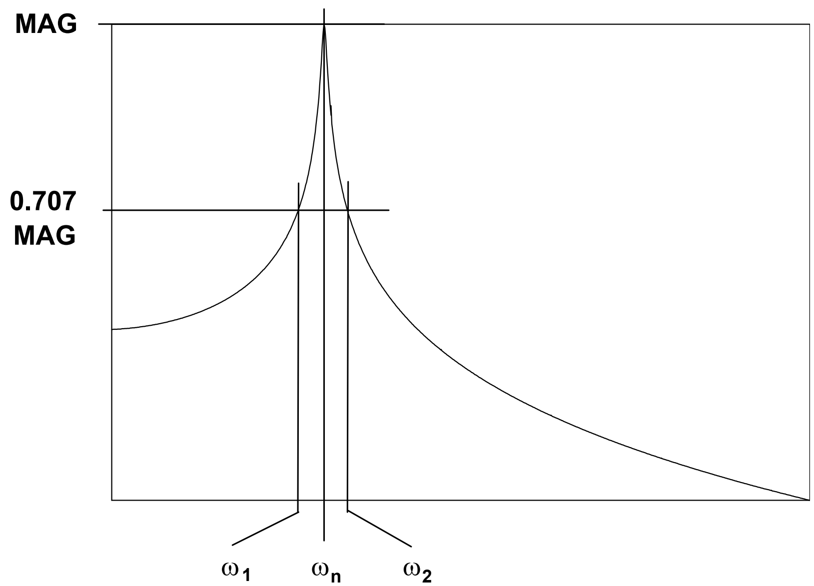


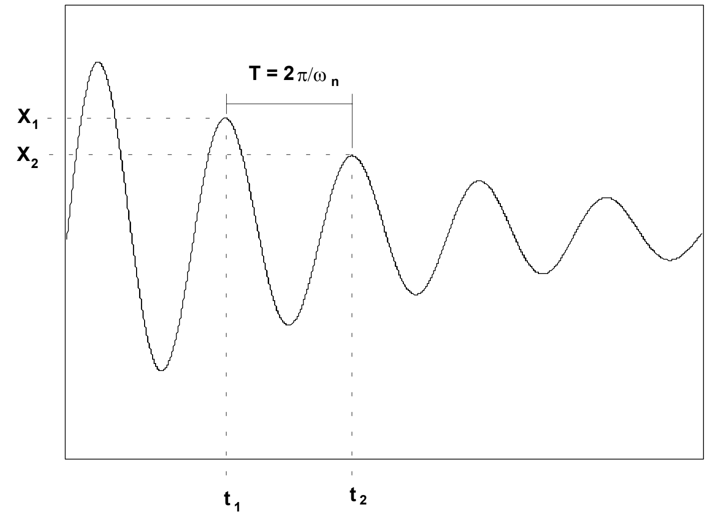


$$\frac{x}{\delta_{st}} = \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\beta}{1-\beta^2}\right)$$

SDOF - Damping Approximations





$$Q = \frac{1}{2\zeta} = \frac{\omega_n}{\omega_2 - \omega_1}$$

$$\delta = \ln\frac{x_1}{x_2} \approx 2\pi\zeta$$

SDOF - Laplace Domain

Equation of Motion in Laplace Domain

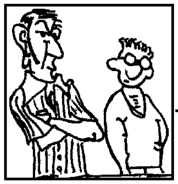
$$(ms^2+cs+k)x(s) = f(s) \quad \textbf{with} \quad b(s) = (ms^2+cs+k)$$

System Characteristic Equation

$$b(s)\,x(s) = f(s) \quad \textbf{and} \quad x(s) = b^{-1}(s)f(s) = h(s)f(s)$$

System Transfer Function

$$h(s) = \frac{1}{ms^2+cs+k}$$



SDOF - Transfer Function

Polynomial Form

$$h(s) = \frac{1}{ms^2 + cs + k}$$

Pole-Zero Form

$$h(s) = \frac{1/m}{(s - p_1)(s - p_1^*)}$$

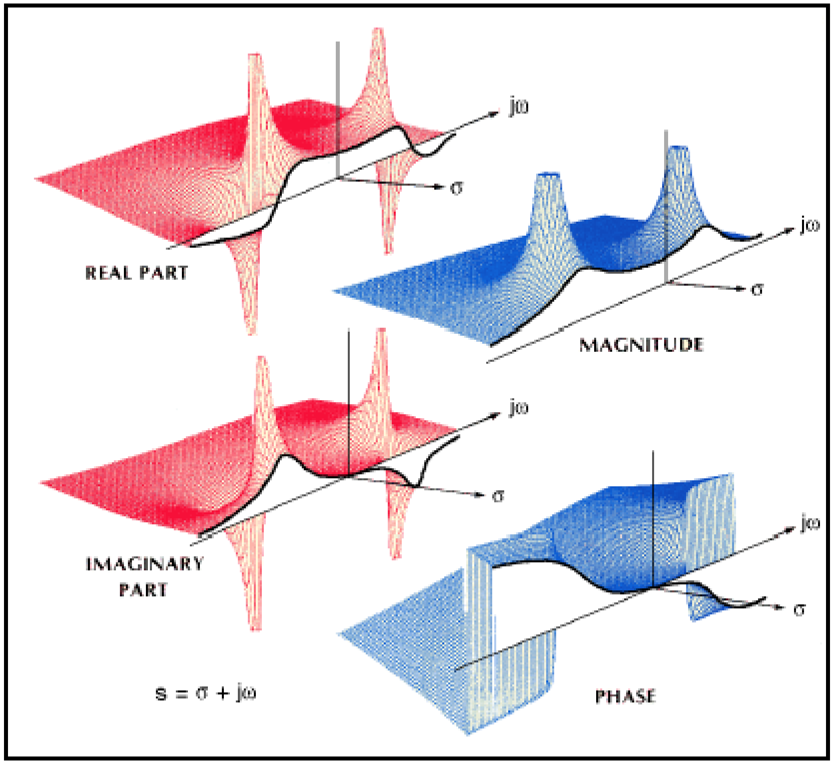
Partial Fraction Form

$$h(s) = \frac{a_1}{(s - p_1)} + \frac{a_1^*}{(s - p_1^*)}$$

Exponential Form

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega t} \sin \omega_d t$$

SDOF - Transfer Function & Residues



Source: Vibrant Technology

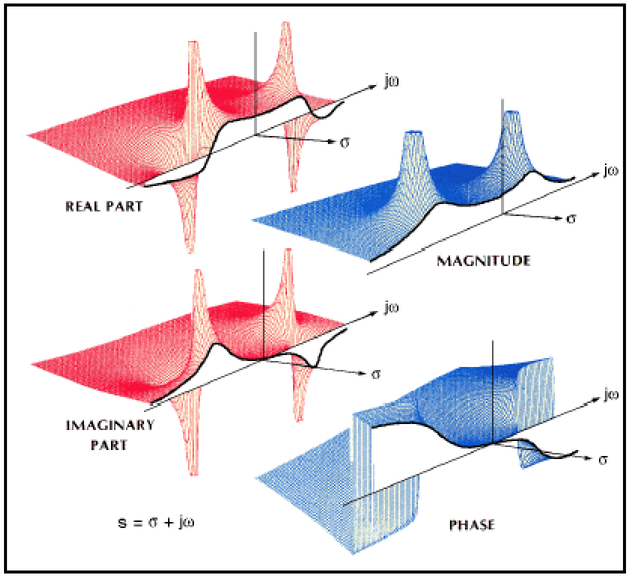
Residue

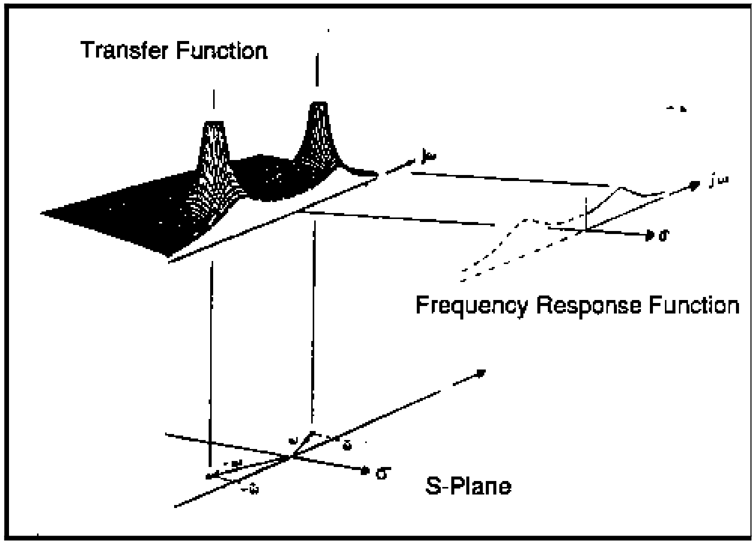
$$a_1 = h(s)(s - p_1)\Big|_{s \to p_1} = \frac{1}{2jm\omega_d}$$

related to mode shapes

SDOF - Frequency Response Function

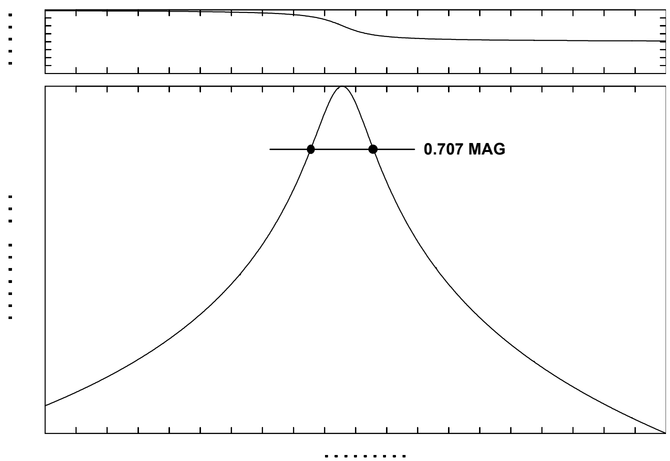
$$h(j\omega) = h(s)\Big|_{s=j\omega} = \frac{a_1}{(j\omega - p_1)} + \frac{a_1^*}{(j\omega - p_1^*)}$$



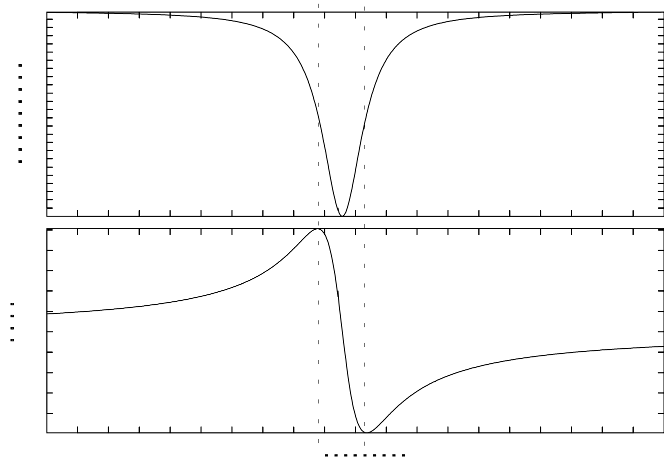


SDOF - Frequency Response Function

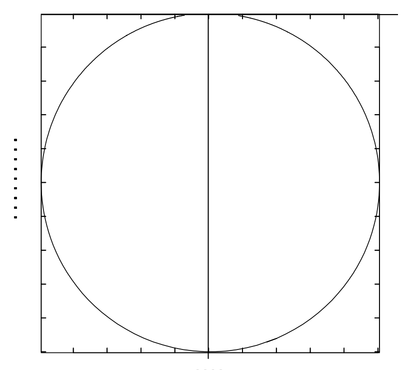
Bode Plot



Coincident-Quadrature Plot



Nyquist Plot

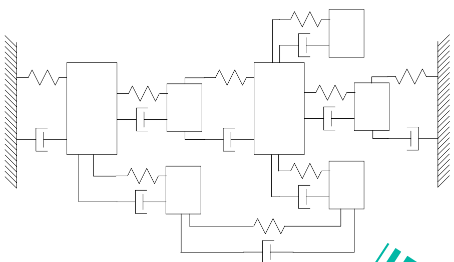


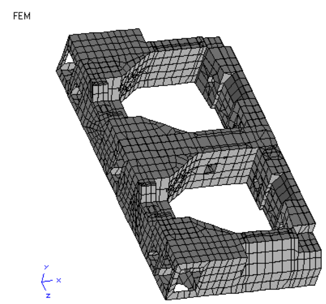
SDOF - Frequency Response Function

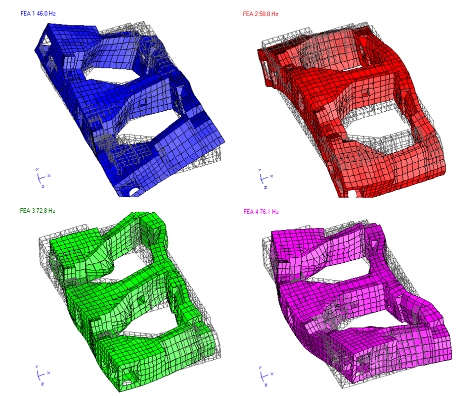
DYNAMIC COMPLIANCE	DISPLACEMENT / FORCE
MOBILITY	VELOCITY / FORCE
INERTANCE	ACCELERATION / FORCE
DYNAMIC STIFFNESS	FORCE / DISPLACEMENT
MECHANICAL IMPEDANCE	FORCE / VELOCITY
DYNAMIC MASS	FORCE / ACCELERATION

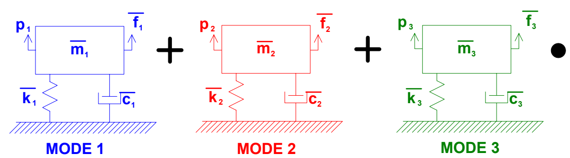
Multiple Degree of Freedom Overview

$$[B(s)]^{-1} = [H(s)] = \frac{Adj[B(s)]}{det[B(s)]} = \frac{[A(s)]}{det[B(s)]}$$

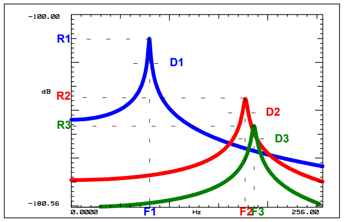


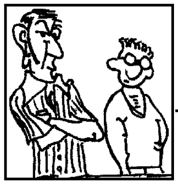






$$\begin{bmatrix} \ddots & & \\ & \overline{M} & \\ & & \ddots \end{bmatrix}\{\ddot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{C} & \\ & & \ddots \end{bmatrix}\{\dot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{K} & \\ & & \ddots \end{bmatrix}\{p\} = [U]^T\{F\}$$

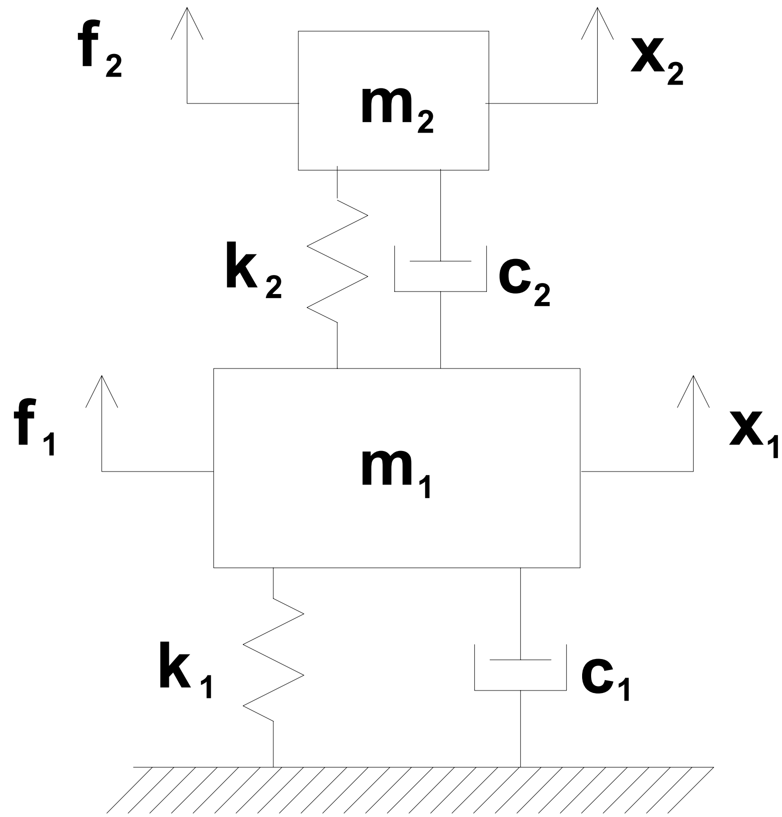




MDOF Definitions

Assumptions

- lumped mass
- stiffness proportional to displacement
- damping proportional to velocity
- linear time invariant
- 2nd order differential equations



MDOF Equations

Equation of Motion - Force Balance

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=f_1(t)$$

$$m_2\ddot{x}_2-c_2\dot{x}_1+c_2\dot{x}_2-k_2x_1+k_2x_2=f_2(t)$$

Matrix Formulation

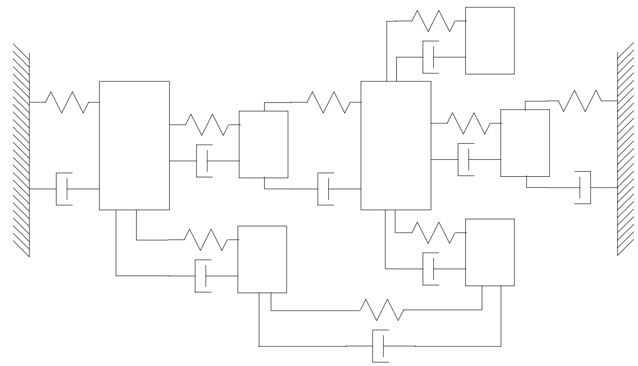
$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Matrices and Linear Algebra are important !!!

MDOF Equations

Equation of Motion

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$



Eigensolution

$$[[K]-\lambda[M]]\{x\}=0$$

Frequencies (eigenvalues) and Mode Shapes (eigenvectors)

$$\begin{bmatrix} \diagdown & & \\ & \Omega^2 & \\ & & \diagdown \end{bmatrix} = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \diagdown \end{bmatrix} \text{ and } [U]=[\{u_1\} \quad \{u_2\} \quad \cdots]$$

Modal Space Transformation

Modal transformation

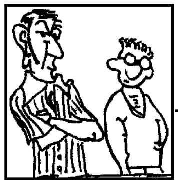
$$\{x\} = [U]\{p\} = [\{u_1\} \quad \{u_2\} \quad \cdots]\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \end{Bmatrix}$$

Projection operation

$$[U]^T[M][U]\{\ddot{p}\} + [U]^T[C][U]\{\dot{p}\} + [U]^T[K][U]\{p\} = [U]^T\{F\}$$

Modal equations (uncoupled)

$$\begin{bmatrix} \bar{m}_1 & & \\ & \bar{m}_2 & \\ & & \ddots \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} \bar{c}_1 & & \\ & \bar{c}_2 & \\ & & \ddots \end{bmatrix}\begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} \bar{k}_1 & & \\ & \bar{k}_2 & \\ & & \ddots \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \{u_1\}^T\{F\} \\ \{u_2\}^T\{F\} \\ \vdots \end{Bmatrix}$$

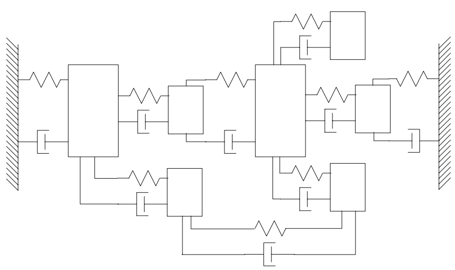


Modal Space Transformation

Diagonal Matrices -

Modal Mass *Modal Damping* *Modal Stiffness*

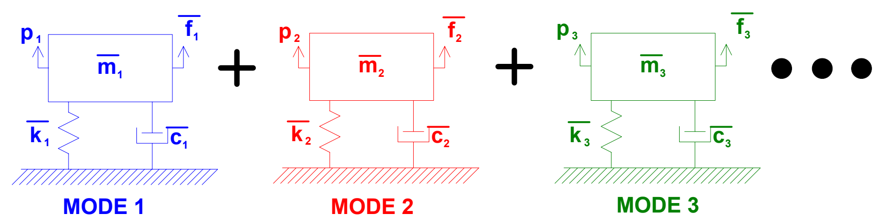
$$\begin{bmatrix} \ddots & & \\ & \overline{M} & \\ & & \ddots \end{bmatrix}\{\ddot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{C} & \\ & & \ddots \end{bmatrix}\{\dot{p}\} + \begin{bmatrix} \ddots & & \\ & \overline{K} & \\ & & \ddots \end{bmatrix}\{p\} = [U]^T\{F\}$$



Highly coupled system

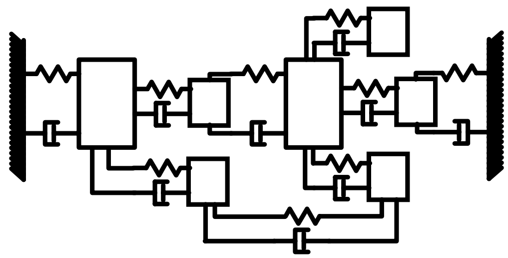
transformed into

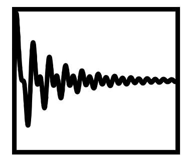
simple system



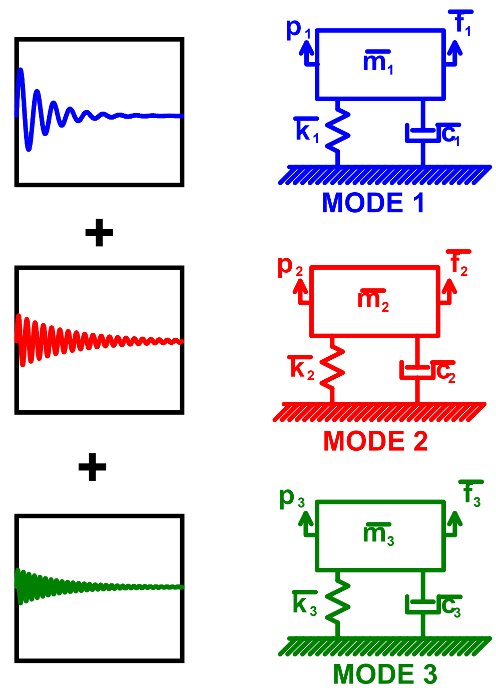
Modal Space Transformation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$





$= \quad \Sigma$



$$\{x\} = [U]\{p\} = [\{u_1\}\{u_2\}\{u_3\} \bullet\bullet\bullet]\{p\}$$

MODAL ⬇ SPACE

$$[\,\diagdown\overline{M}\diagdown\,]\{\ddot{p}\} + [\,\diagdown\overline{C}\diagdown\,]\{\dot{p}\} + [\,\diagdown\overline{K}\diagdown\,]\{p\} = [U]^T\{\overline{F}(t)\}$$

$$\{u_1\}p_1 + \{u_2\}p_2 + \{u_3\}p_3$$

MDOF - Laplace Domain

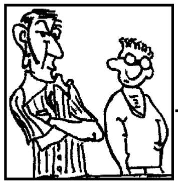
Laplace Domain Equation of Motion

$$\left[[M]s^2 + [C]s + [K]\right]\{x(s)\} = 0 \Rightarrow [B(s)]\{x(s)\} = 0$$

System Characteristic (Homogeneous) Equation

$$\left|[M]s^2 + [C]s + [K]\right| = 0 \Rightarrow p_k = -\sigma_k \pm j\omega_{dk}$$

Damping **Frequency**



MDOF - Transfer Function

System Equation

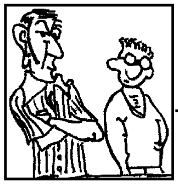
$$[B(s)]\{x(s)\} = \{F(s)\} \;\Rightarrow\; [H(s)] = [B(s)]^{-1} = \frac{\{x(s)\}}{\{F(s)\}}$$

System Transfer Function

$$[B(s)]^{-1} = [H(s)] = \frac{\mathrm{Adj}[B(s)]}{\det[B(s)]} = \frac{[A(s)]}{\det[B(s)]}$$

$[A(s)]$ ***Residue Matrix*** → ***Mode Shapes***

$\det[B(s)]$ ***Characteristic Equation*** → ***Poles***



MDOF - Residue Matrix and Mode Shapes

Transfer Function evaluated at one pole

$$\left[H(s)\right]_{s=s_k} = \{u_k\} \frac{q_k}{s-p_k} \{u_k\}^T$$

can be expanded for all modes

$$\left[H(s)\right] = \sum_{k=1}^{m} \frac{q_k \{u_k\}\{u_k\}^T}{(s-p_k)} + \frac{q_k \{u_k^*\}\{u_k^*\}^T}{\left(s-p_k^*\right)}$$

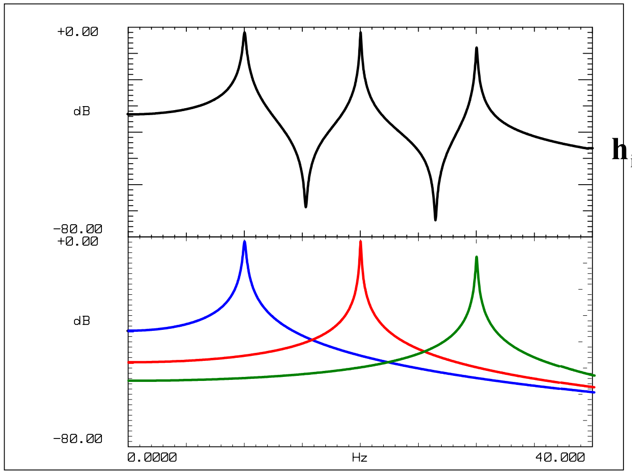
MDOF - Residue Matrix and Mode Shapes

Residues are related to mode shapes as

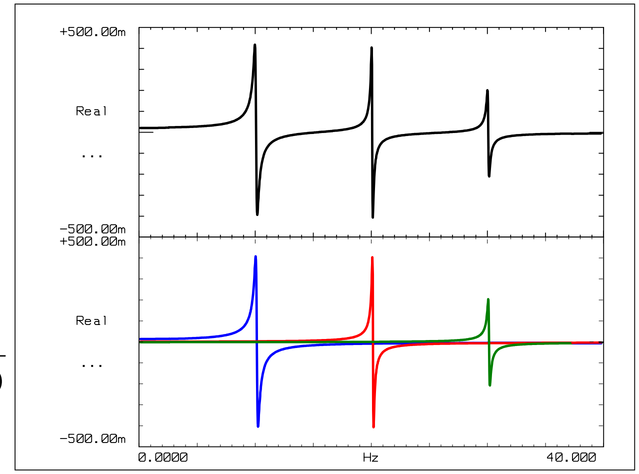
$$[A(s)]_k = q_k \{u_k\}\{u_k\}^T$$

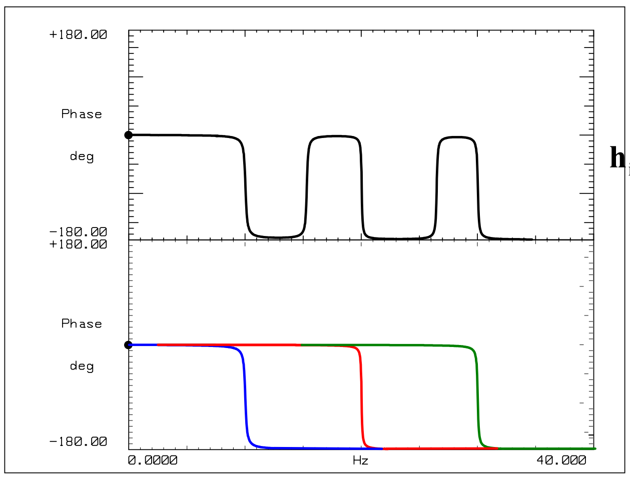
$$\begin{bmatrix} a_{11k} & a_{12k} & a_{13k} & \cdots \\ a_{21k} & a_{22k} & a_{23k} & \cdots \\ a_{31k} & a_{32k} & a_{33k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = q_k \begin{bmatrix} u_{1k}u_{1k} & u_{1k}u_{2k} & u_{1k}u_{3k} & \cdots \\ u_{2k}u_{1k} & u_{2k}u_{2k} & u_{2k}u_{3k} & \cdots \\ u_{3k}u_{1k} & u_{3k}u_{2k} & u_{3k}u_{3k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

MDOF - Drive Point FRF

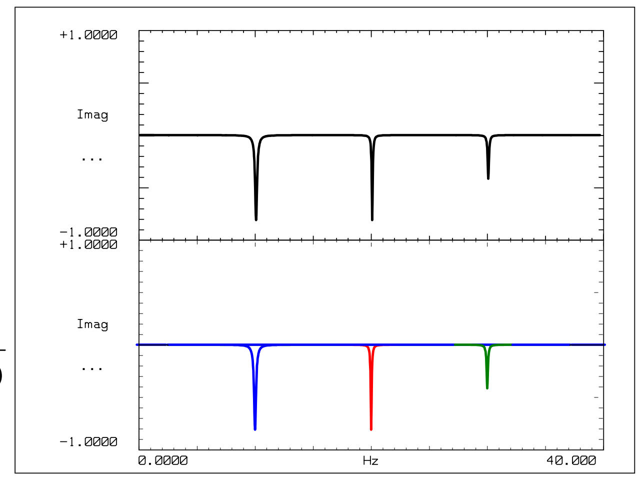


$$h_{ij}(j\omega)=\frac{a_{ij1}}{(j\omega-p_1)}+\frac{a_{ij1}^*}{(j\omega-p_1^*)}+\frac{a_{ij2}}{(j\omega-p_2)}+\frac{a_{ij2}^*}{(j\omega-p_2^*)}+\frac{a_{ij3}}{(j\omega-p_3)}+\frac{a_{ij3}^*}{(j\omega-p_3^*)}$$

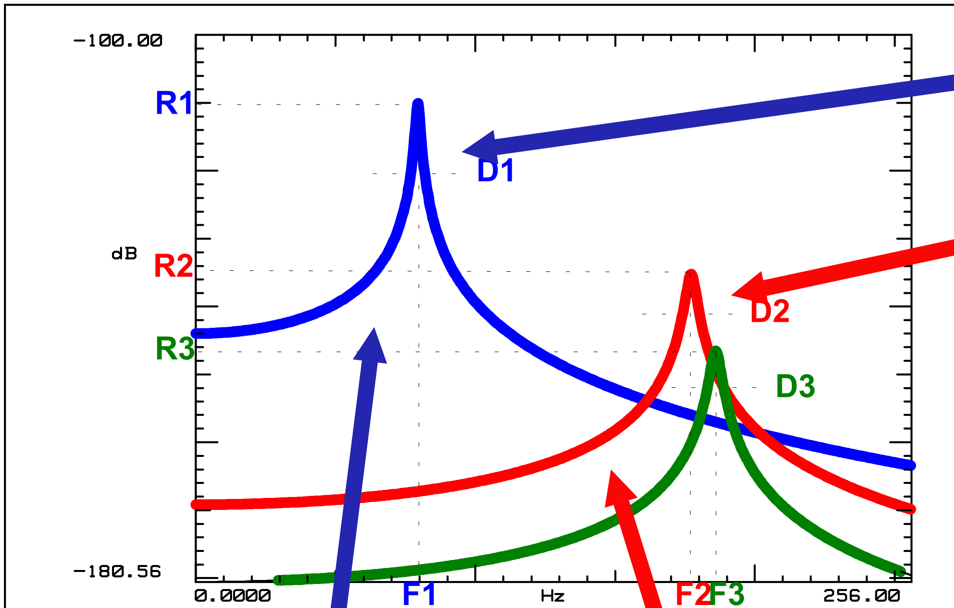




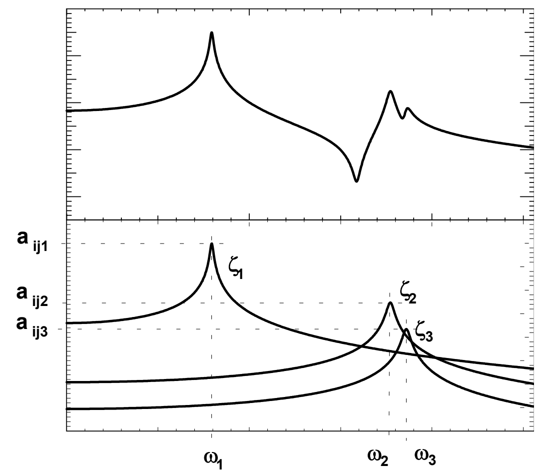
$$h_{ij}(j\omega)=\frac{q_1u_{i1}u_{j1}}{(j\omega-p_1)}+\frac{q_1u_{i1}u_{j1}^*}{(j\omega-p_1^*)}+\frac{q_2u_{i2}u_{j2}}{(j\omega-p_2)}+\frac{q_2u_{i2}u_{j2}^*}{(j\omega-p_2^*)}+\frac{q_3u_{i3}u_{j3}}{(j\omega-p_3)}+\frac{q_3u_{i3}u_{j3}^*}{(j\omega-p_3^*)}$$



MDOF - FRF using Residues or Mode Shapes

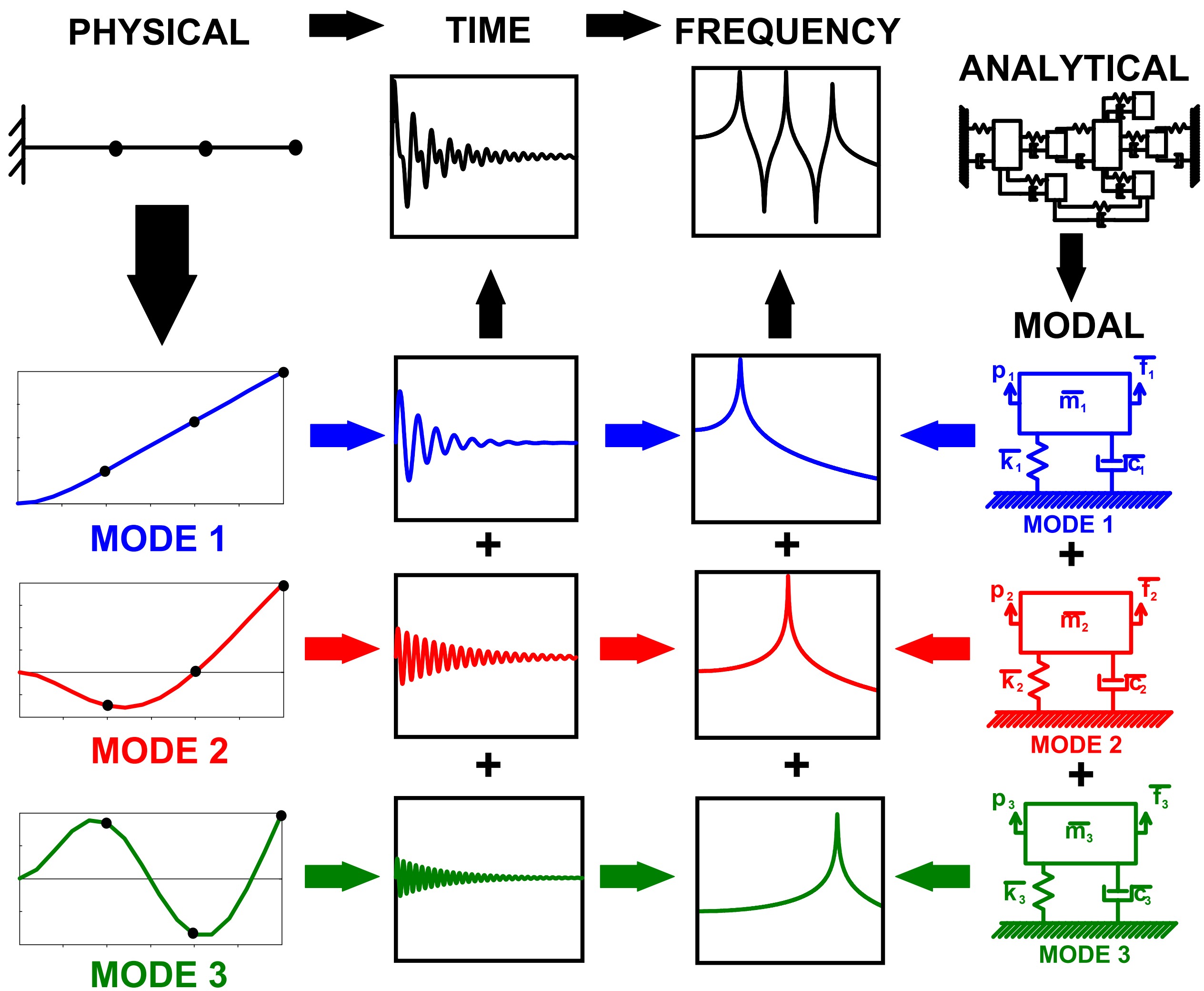


$$h_{ij}(j\omega) = \frac{a_{ij1}}{(j\omega - p_1)} + \frac{a^*_{ij1}}{(j\omega - p^*_1)} + \frac{a_{ij2}}{(j\omega - p_2)} + \frac{a^*_{ij2}}{(j\omega - p^*_2)} + \cdots$$

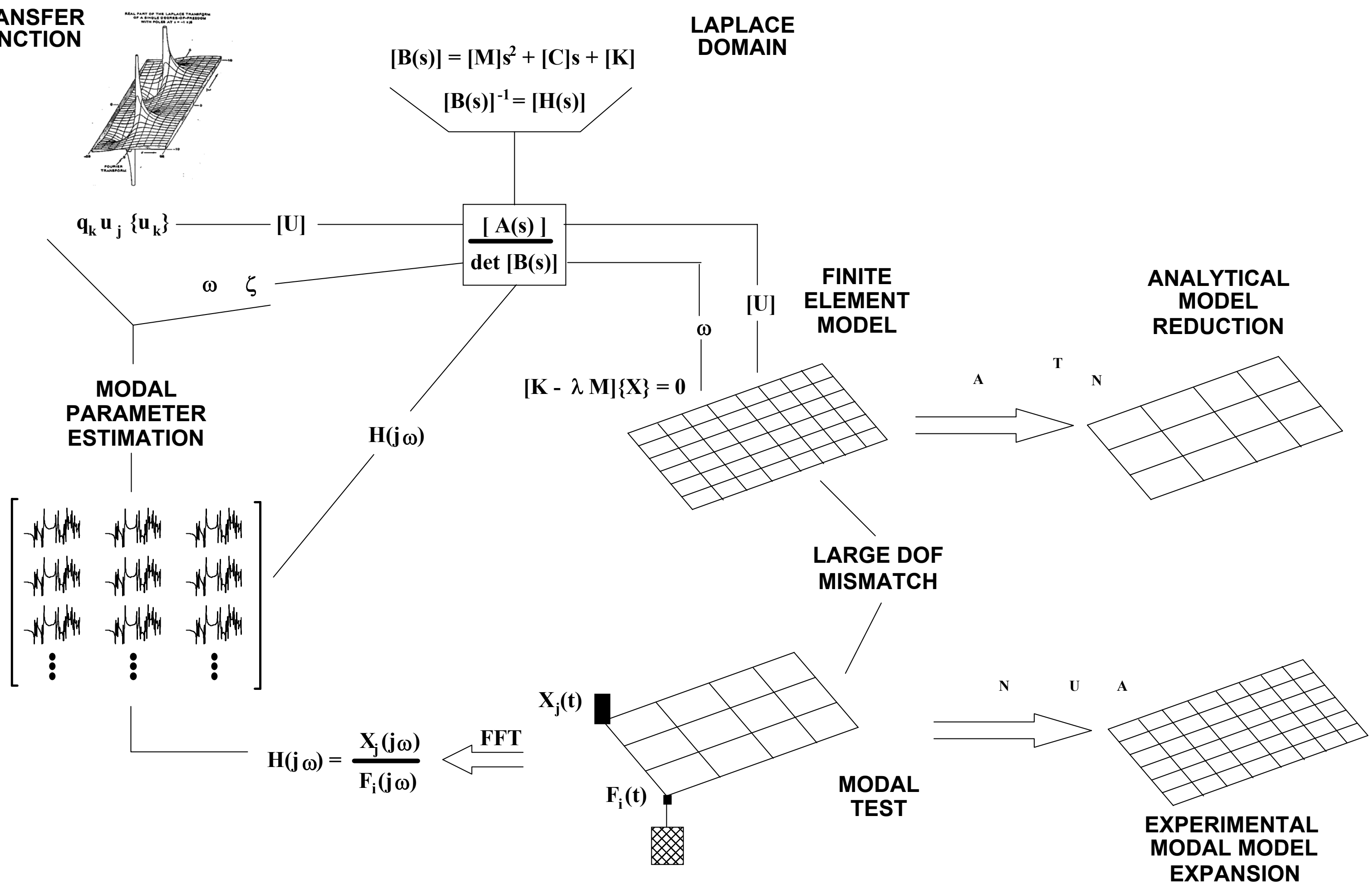


$$h_{ij}(j\omega) = \frac{q_1 u_{i1} u_{j1}}{(j\omega - p_1)} + \frac{q^*_1 u^*_{i1} u^*_{j1}}{(j\omega - p^*_1)} + \frac{q_2 u_{i2} u_{j2}}{(j\omega - p_2)} + \frac{q^*_2 u^*_{i2} u^*_{j2}}{(j\omega - p^*_2)} + \cdots$$

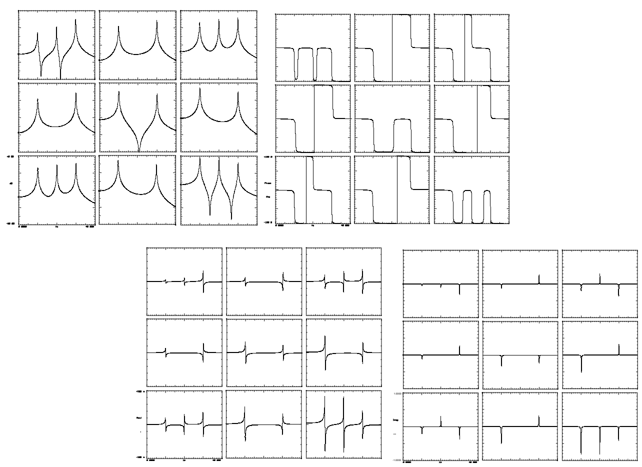
Time / Frequency / Modal Representation

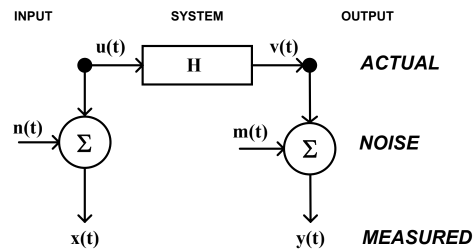


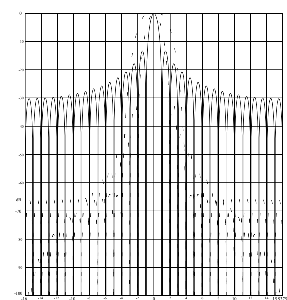
Overview Analytical and Experimental Modal Analysis



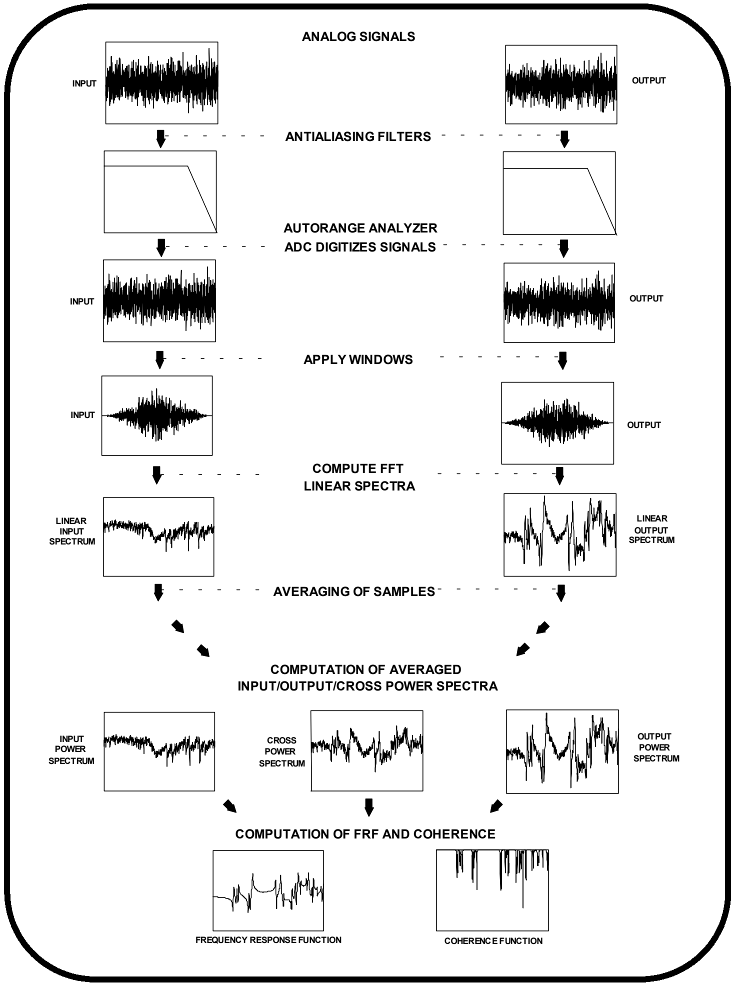
Measurement Definitions







Measurement Definitions



Actual time signals

Analog anti-alias filter

Digitized time signals

Windowed time signals

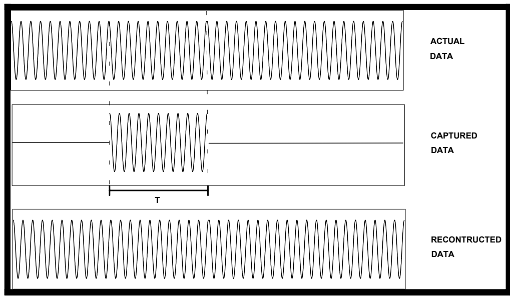
Compute FFT of signal

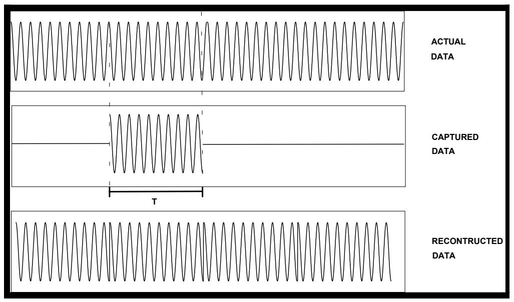
Average auto/cross spectra

Compute FRF and Coherence

Leakage

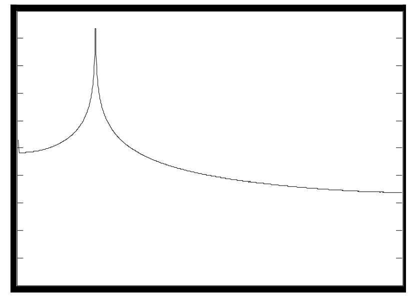
T
I
M
E



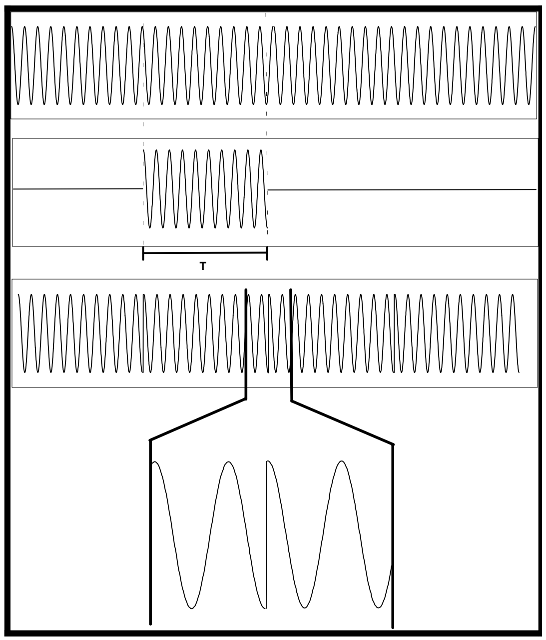


Periodic Signal

Non-Periodic Signal

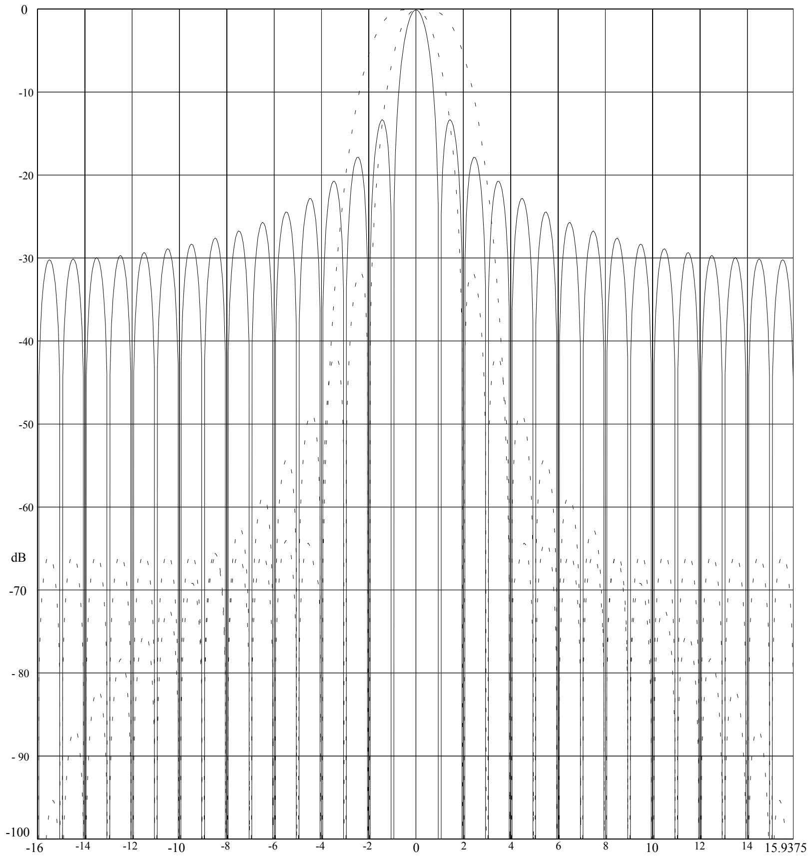


F
R
E
Q
U
E
N
C
Y



Leakage due to signal distortion

Windows



Time weighting functions are applied to minimize the effects of leakage

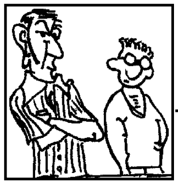
Rectangular

Hanning

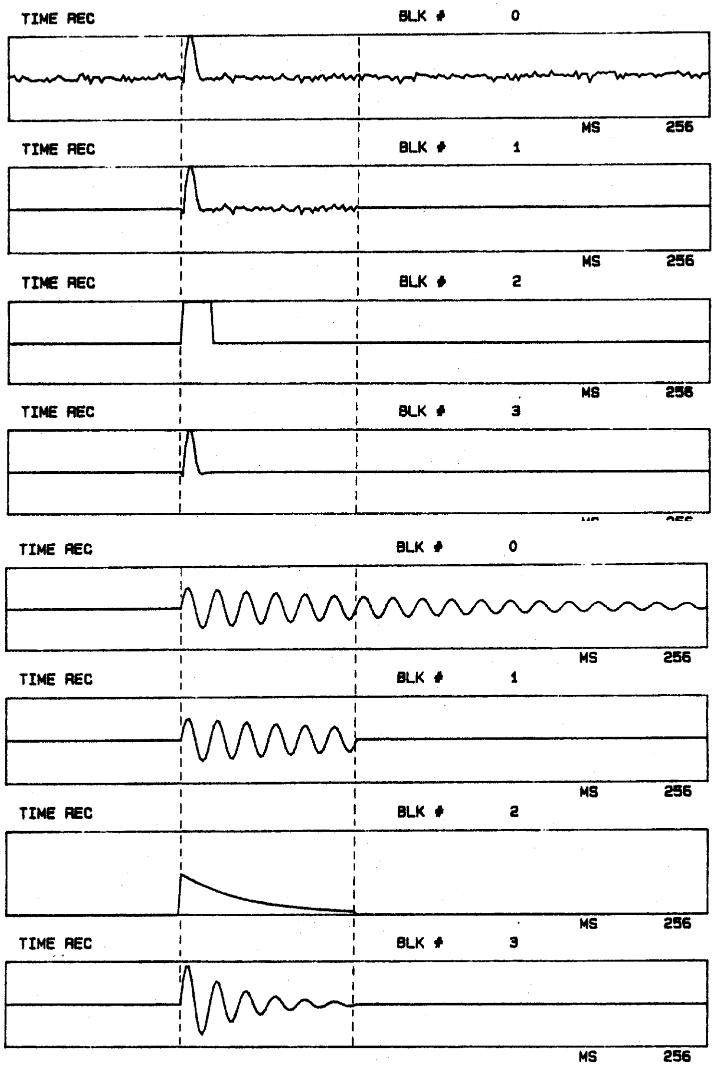
Flat Top

and many others

Windows DO NOT eliminate leakage !!!



Windows

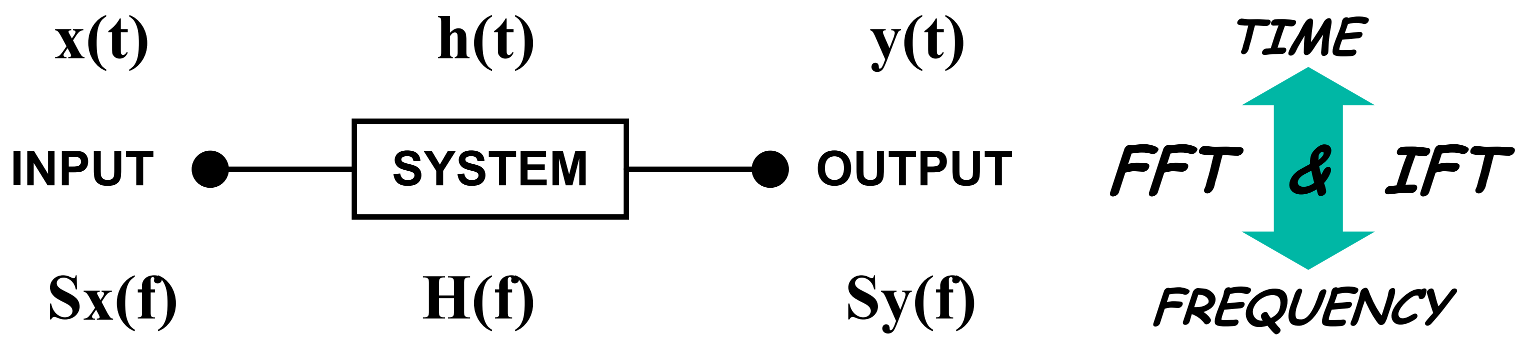


Special windows are used for impact testing

Force window

Exponential Window

Measurements - Linear Spectra



x(t)	- time domain input to the system
y(t)	- time domain output to the system
Sx(f)	- linear Fourier spectrum of x(t)
Sy(f)	- linear Fourier spectrum of y(t)
H(f)	- system transfer function
h(t)	- system impulse response

Measurements - Linear Spectra

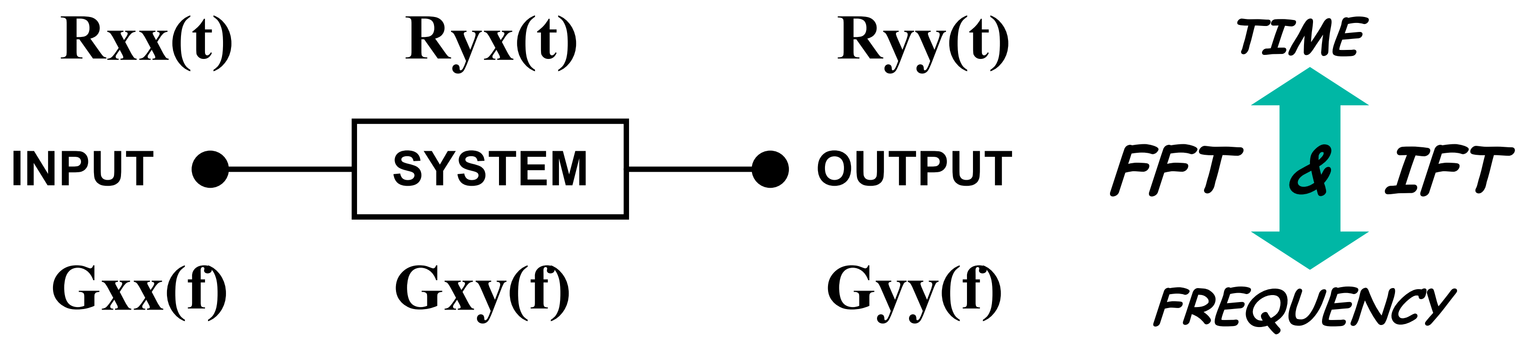
$$x(t)=\int_{-\infty}^{+\infty} S_x(f)e^{j2\pi ft}df \qquad S_x(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$

$$y(t)=\int_{-\infty}^{+\infty} S_y(f)e^{j2\pi ft}df \qquad S_y(f)=\int_{-\infty}^{+\infty} y(t)e^{-j2\pi ft}dt$$

$$h(t)=\int_{-\infty}^{+\infty} H(f)e^{j2\pi ft}df \qquad H(f)=\int_{-\infty}^{+\infty} h(t)e^{-j2\pi ft}dt$$

Note: S_X and S_Y are complex valued functions

Measurements - Power Spectra



$R_{xx}(t)$ - autocorrelation of the input signal $x(t)$

$R_{yy}(t)$ - autocorrelation of the output signal $y(t)$

$R_{yx}(t)$ - cross correlation of $y(t)$ and $x(t)$

$G_{xx}(f)$ - autopower spectrum of $x(t)$ $\quad G_{xx}(f) = S_x(f) \bullet S_x^*(f)$

$G_{yy}(f)$ - autopower spectrum of $y(t)$ $\quad G_{yy}(f) = S_y(f) \bullet S_y^*(f)$

$G_{yx}(f)$ - cross power spectrum of $y(t)$ and $x(t)$ $\quad G_{yx}(f) = S_y(f) \bullet S_x^*(f)$

Measurements - Linear Spectra

$$R_{xx}(\tau)=E[x(t),x(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T x(t)x(t+\tau)dt$$

$$G_{xx}(f)=\int_{-\infty}^{+\infty}R_{xx}(\tau)e^{-j2\pi ft}d\tau=S_x(f)\bullet S_x^*(f)$$

$$R_{yy}(\tau)=E[y(t),y(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T y(t)y(t+\tau)dt$$

$$G_{yy}(f)=\int_{-\infty}^{+\infty}R_{yy}(\tau)e^{-j2\pi ft}d\tau=S_y(f)\bullet S_y^*(f)$$

$$R_{yx}(\tau)=E[y(t),x(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T y(t)x(t+\tau)dt$$

$$G_{yx}(f)=\int_{-\infty}^{+\infty}R_{yx}(\tau)e^{-j2\pi ft}d\tau=S_y(f)\bullet S_x^*(f)$$

Measurements - Derived Relationships

$$S_y = H S_x$$

H1 formulation

- susceptible to noise on the input
- underestimates the actual H of the system

$$S_y \bullet S_x^* = H S_x \bullet S_x^* \qquad H = \frac{S_y \bullet S_x^*}{S_x \bullet S_x^*} = \frac{G_{yx}}{G_{xx}}$$

H2 formulation

- susceptible to noise on the output
- overestimates the actual H of the system

$$S_y \bullet S_y^* = H S_x \bullet S_y^* \qquad H = \frac{S_y \bullet S_y^*}{S_x \bullet S_y^*} = \frac{G_{yy}}{G_{xy}}$$

Other formulations for H exist

COHERENCE

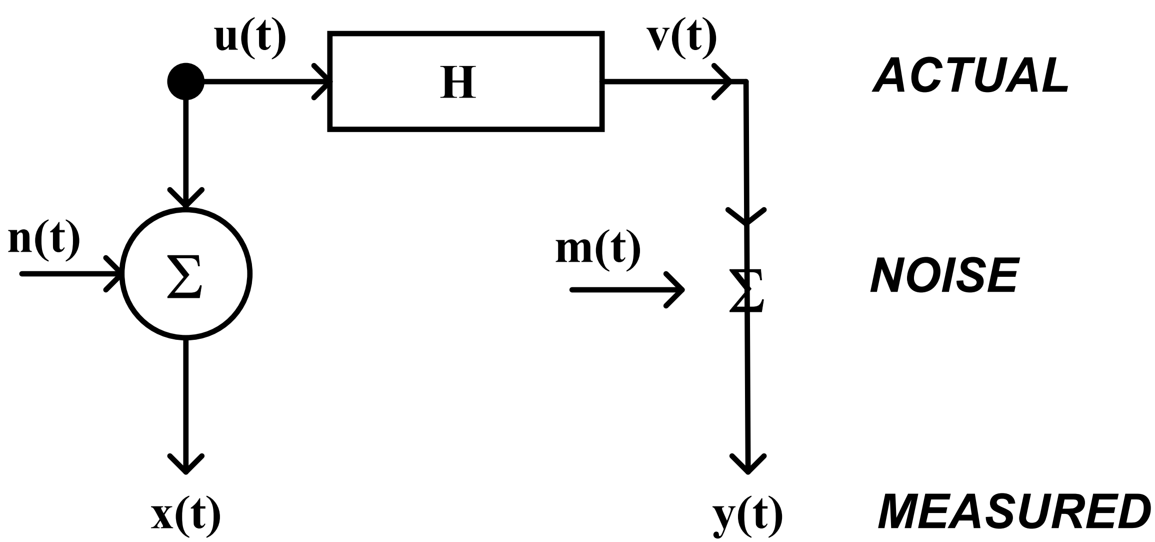
$$\gamma_{xy}^2 = \frac{(S_y \bullet S_x^*)(S_x \bullet S_y^*)}{(S_x \bullet S_x^*)(S_y \bullet S_y^*)} = \frac{G_{yx} / G_{xx}}{G_{yy} / G_{xy}} = \frac{H_1}{H_2}$$

Measurements - Noise

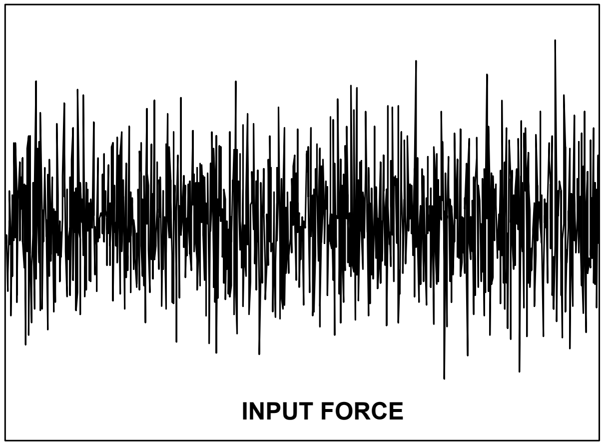
$$H = G_{uv} / G_{uu}$$

$$H_1 = H \left[\frac{1}{1 + \frac{G_{nn}}{G_{uu}}} \right]$$

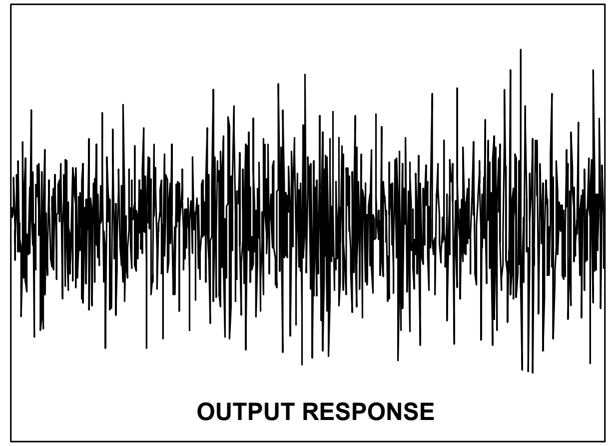
$$H_2 = H \left[1 + \frac{G_{mm}}{G_{vv}} \right]$$

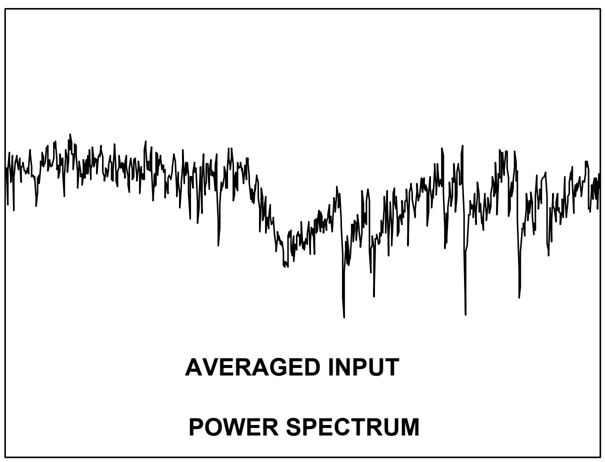


Measurements - Auto Power Spectrum

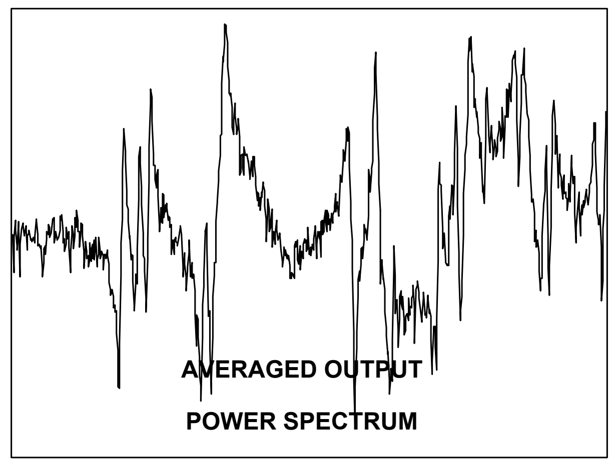


x(t)



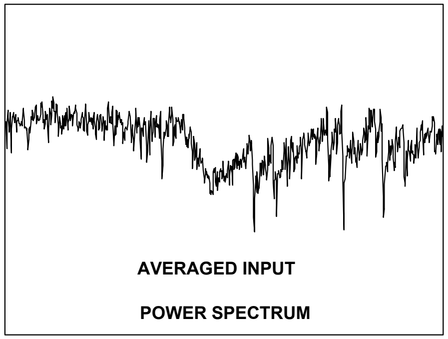


$G_{xx}(f)$

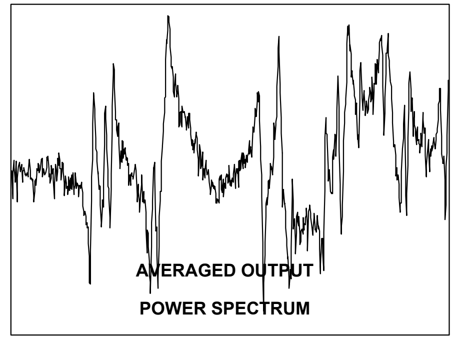


yy

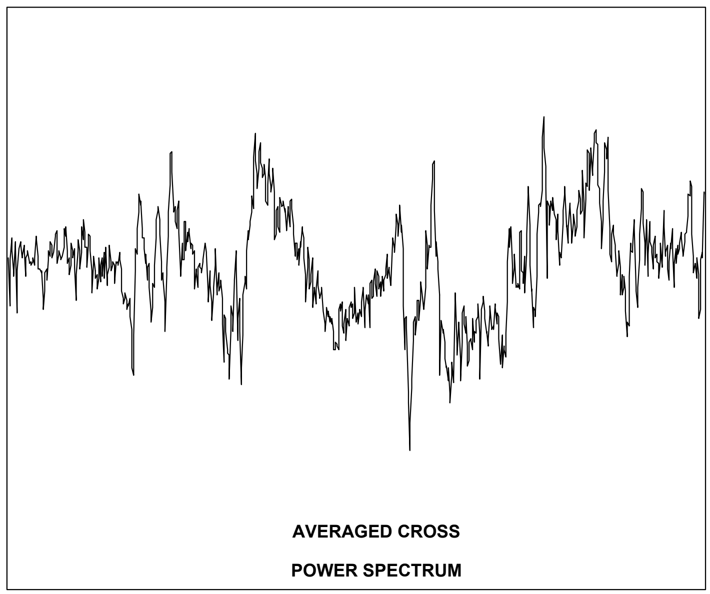
Measurements - Cross Power Spectrum



$G_{xx}(f)$

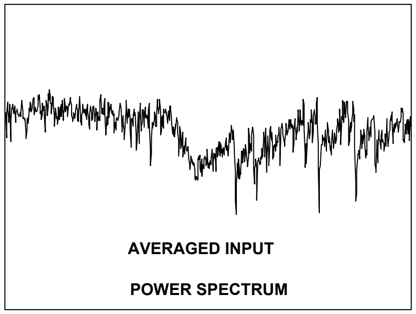


$G_{yy}(f)$

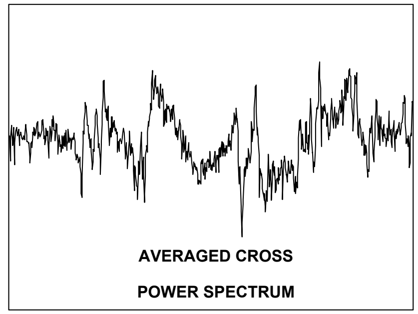


$G_{yx}(f)$

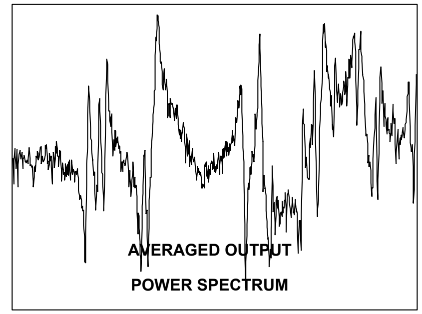
Measurements - Frequency Response Function



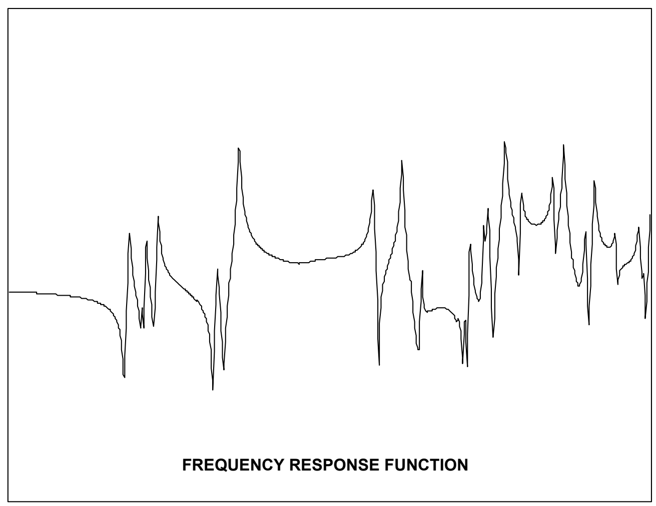
$G_{xx}(f)$



$G_{yx}(f)$

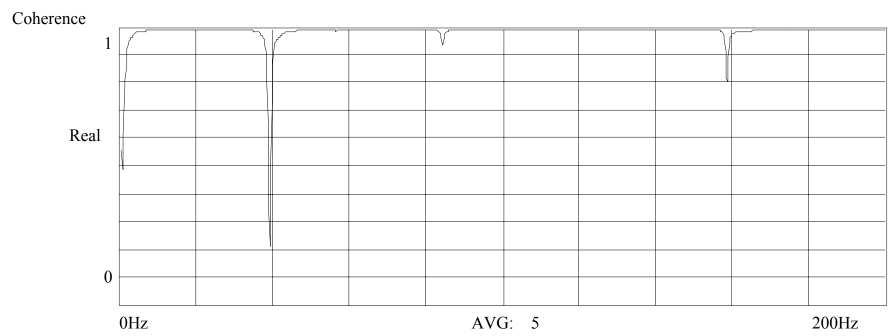


$G_{yy}(f)$

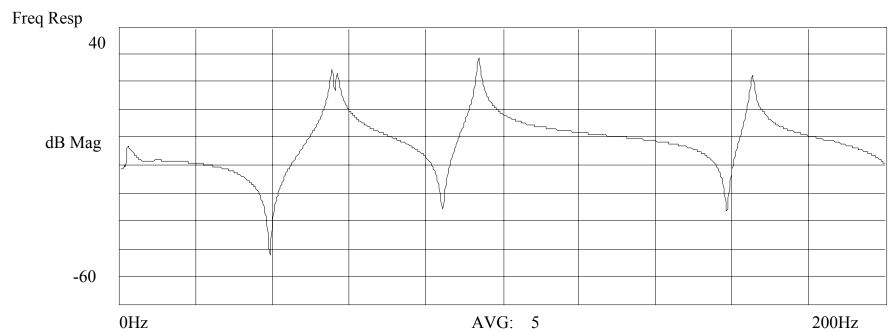


$H(f)$

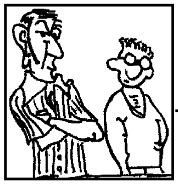
Measurements - FRF & Coherence



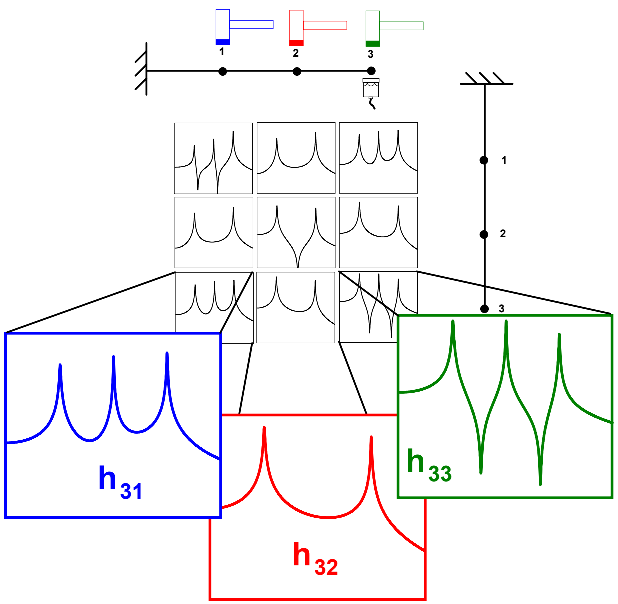
COHERENCE

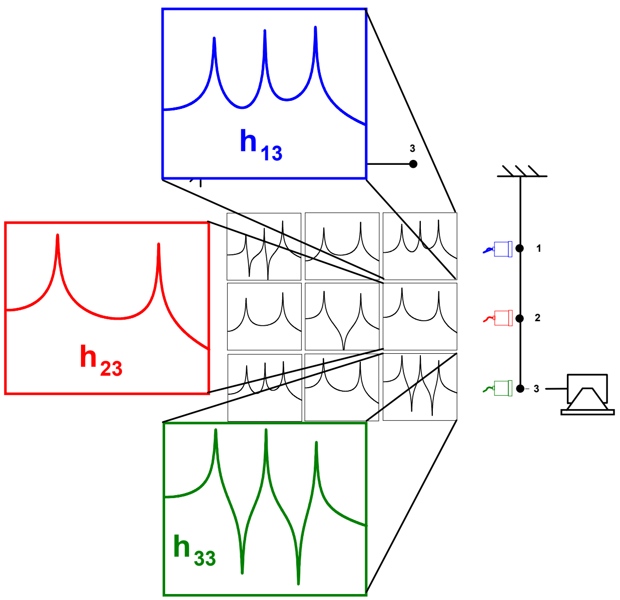


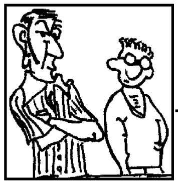
FREQUENCY RESPONSE FUNCTION



Excitation Considerations

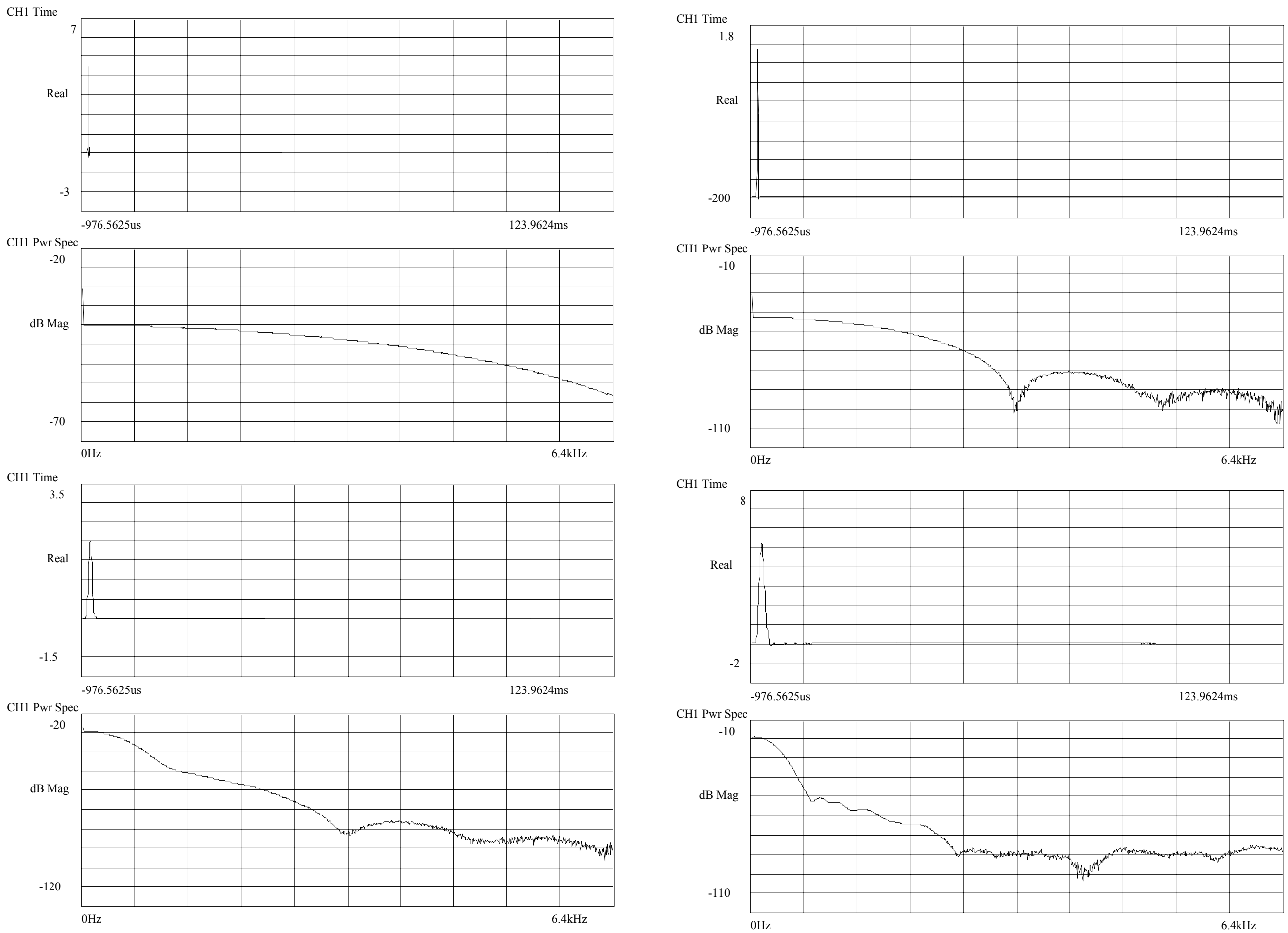






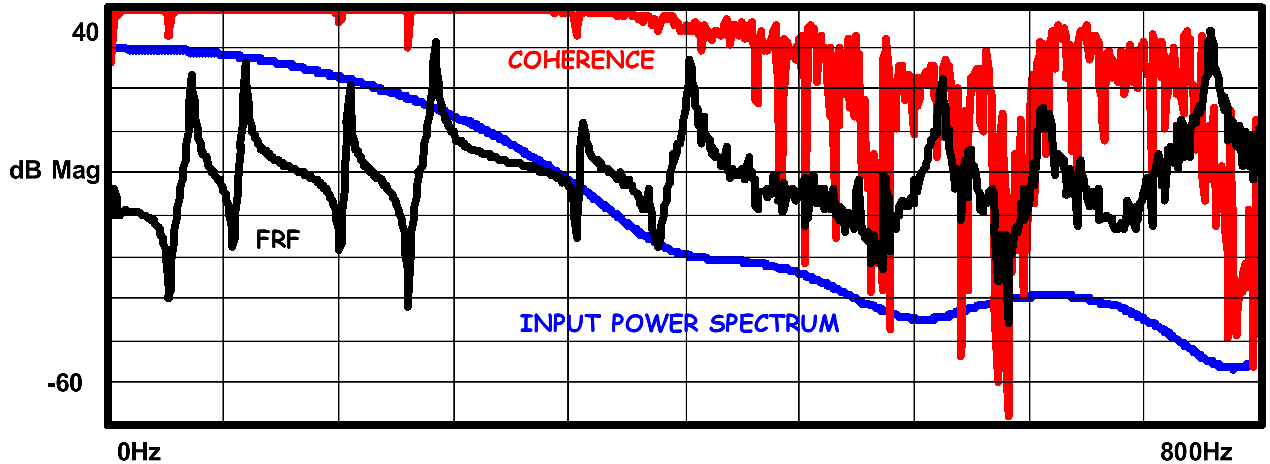
Excitation Considerations - Impact

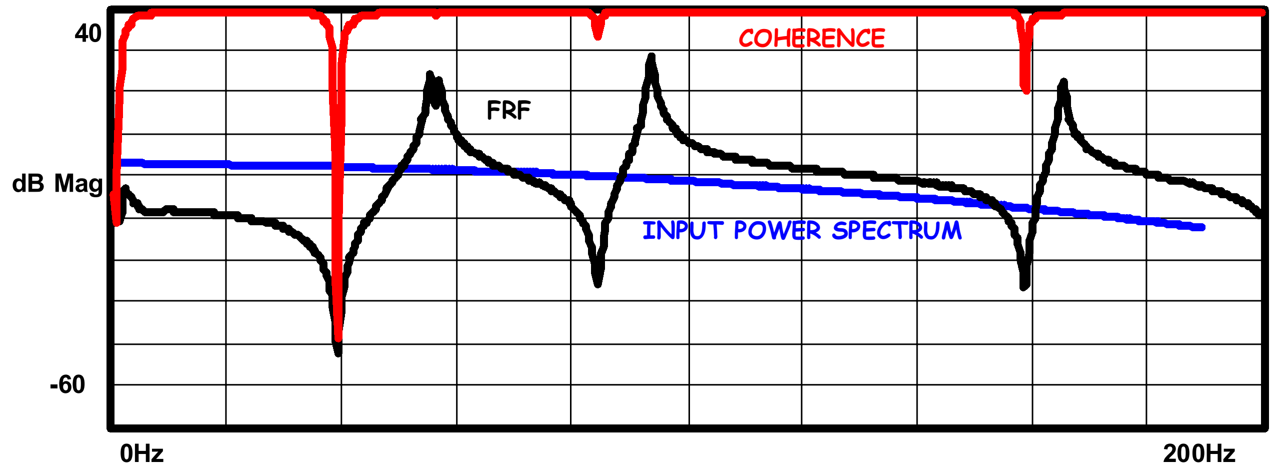
The force spectrum can be customized to some extent through the use of hammer tips with various hardnesses.

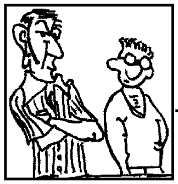


Excitation Considerations - Impact/Exponential

The excitation must be sufficient to excite all the modes of interest over the desired frequency range.

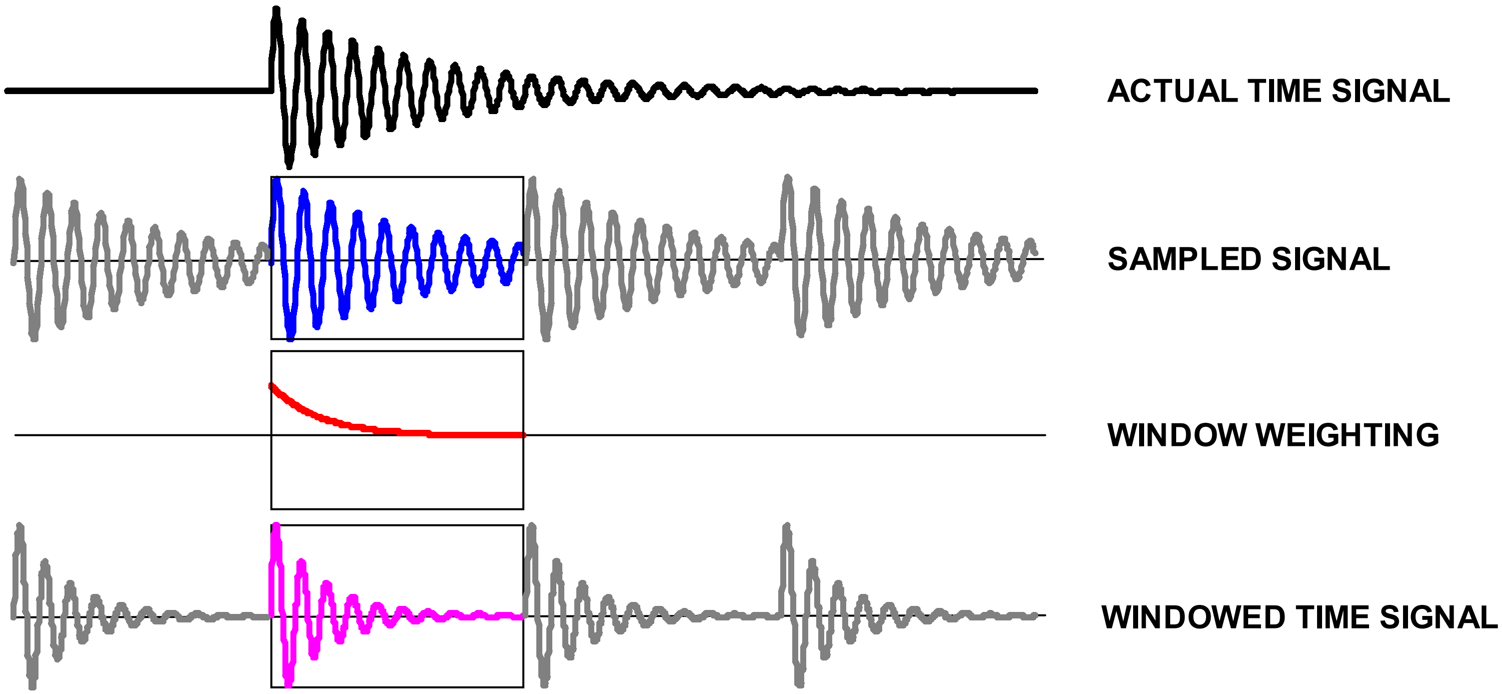


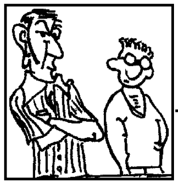




Excitation Considerations - Impact/Exponential

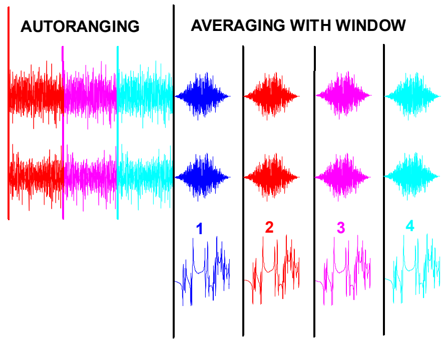
The response due to impact excitation may need an exponential window if leakage is a concern.





Excitation Considerations - Shaker Excitation

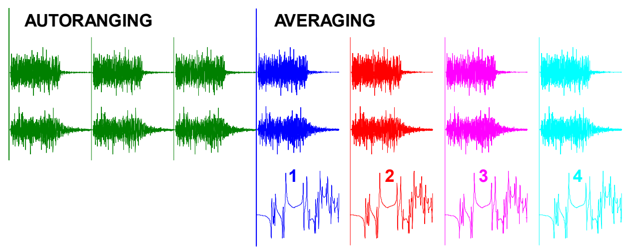
Random with Hanning



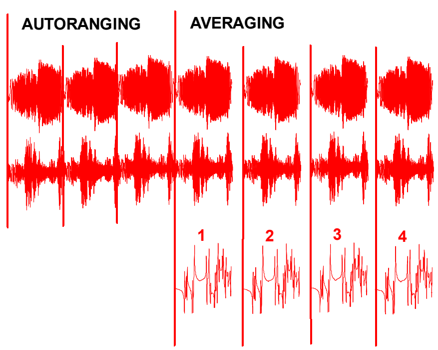
Leakage is a serious concern

Accurate FRFs are necessary

Burst Random



Sine Chirp

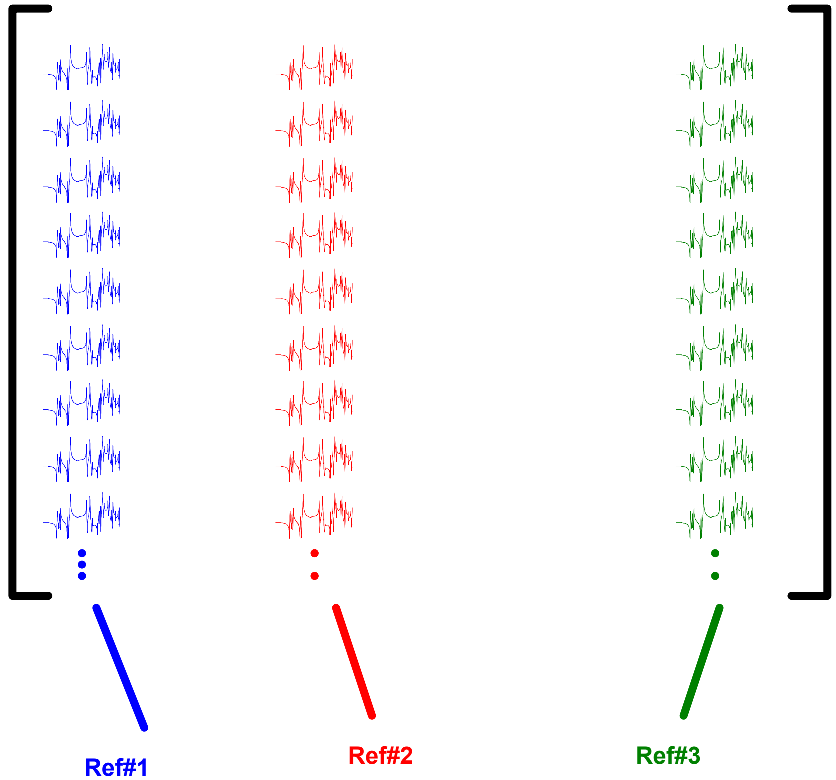


Special excitation techniques can be used which will result in leakage free measurements without the use of a window

as well as other techniques

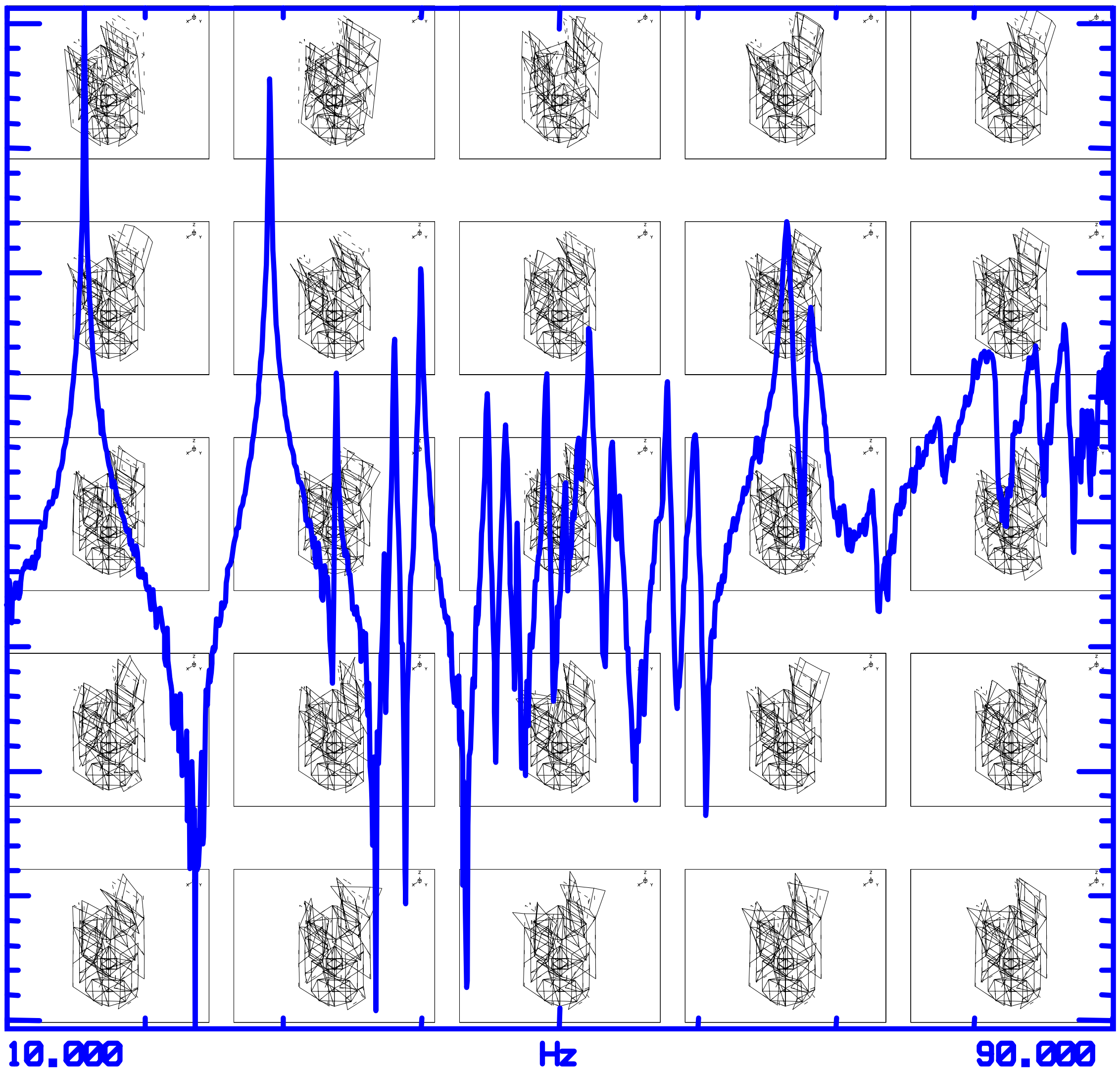
Excitation Considerations - MIMO

Multiple referenced FRFs are obtained from MIMO test

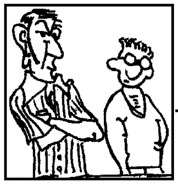


Energy is distributed better throughout the structure making better measurements possible

Excitation Considerations - MIMO



Large or complicated structures require special attention



Excitation Considerations - MIMO

$$[G_{XF}] = [H][G_{FF}]$$

$$[H] = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1,Ni} \\ H_{21} & H_{22} & \cdots & H_{2,Ni} \\ \vdots & \vdots & & \vdots \\ H_{No,1} & H_{No,2} & \cdots & H_{No,Ni} \end{bmatrix}$$

where

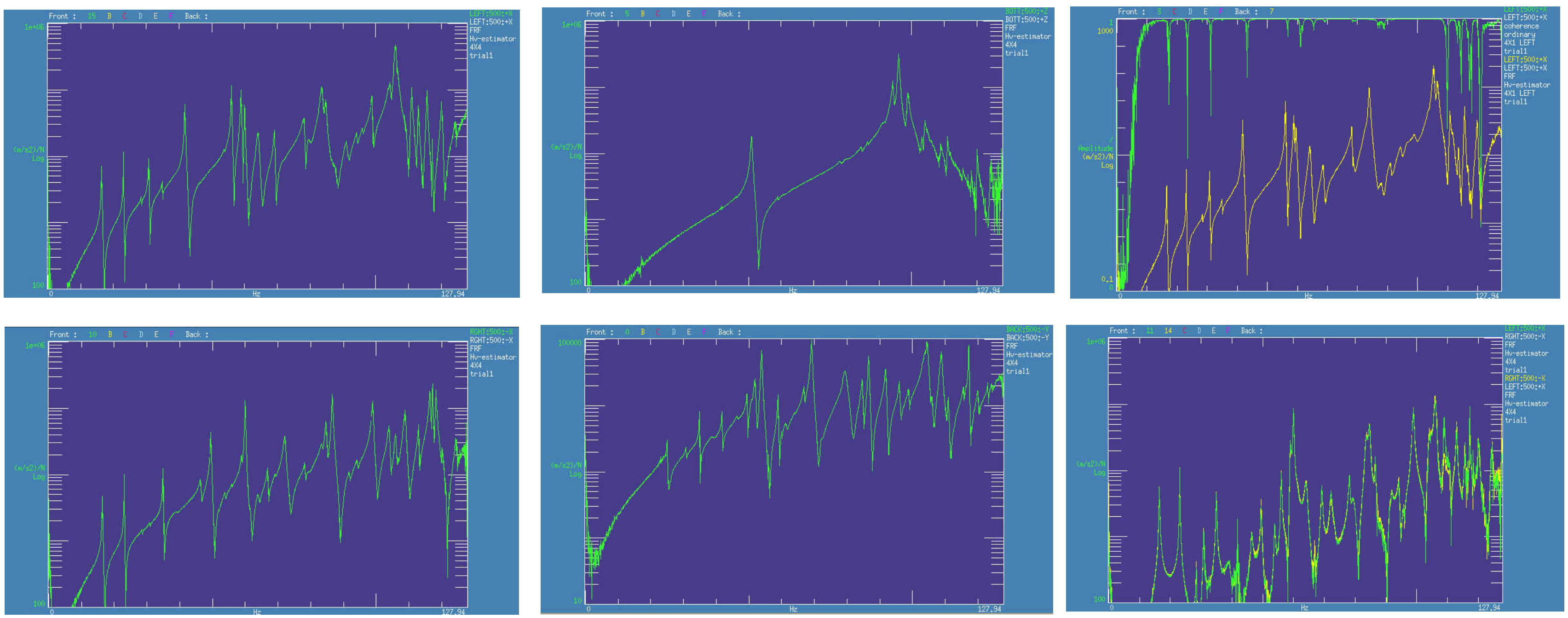
$$[H] = [G_{XF}][G_{FF}]^{-1}$$

No - number of outputs
Ni - number of inputs

Measurements are developed in a similar fashion to the single input single output case but using a matrix formulation

Excitation Considerations - MIMO

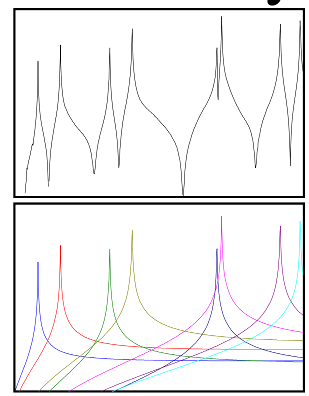
Measurements on the same structure can show tremendously different modal densities depending on the location of the measurement

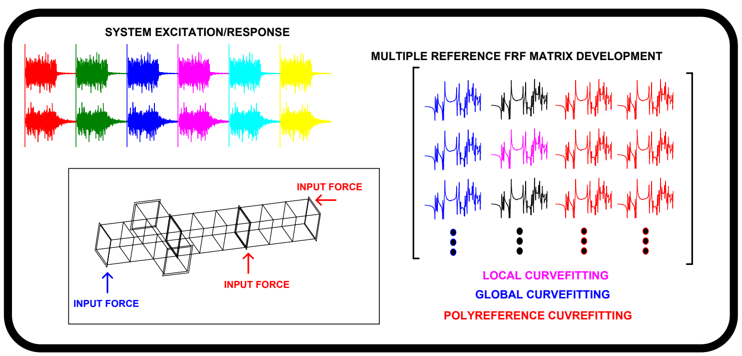


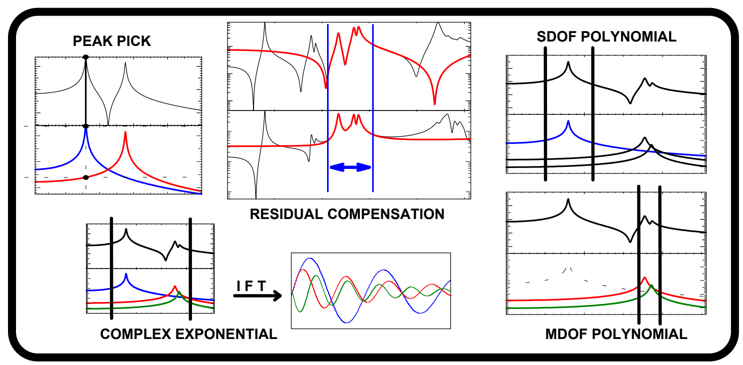
Source: Michigan Technological University Dynamic Systems Laboratory

Modal Parameter Estimation Concepts

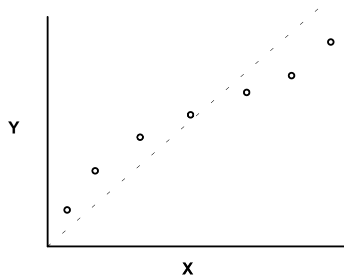
$$[H(s)] = \sum_{\text{terms}}^{\text{lower}} \frac{[A_k]}{(s-s_k)} + \frac{[A_k^*]}{(s-s_k^*)} + \sum_{k=i}^{j} \frac{[A_k]}{(s-s_k)} + \frac{[A_k^*]}{(s-s_k^*)} + \sum_{\text{terms}}^{\text{upper}} \frac{[A_k]}{(s-s_k)} + \frac{[A_k^*]}{(s-s_k^*)}$$





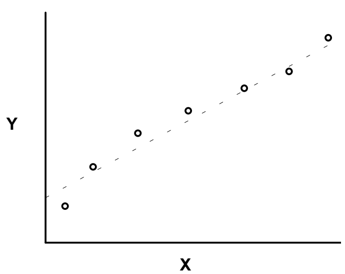


Parameter Estimation Concepts



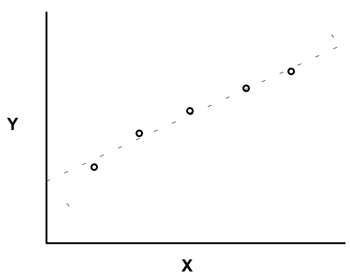
NO COMPENSATION

$$y = m\,x$$

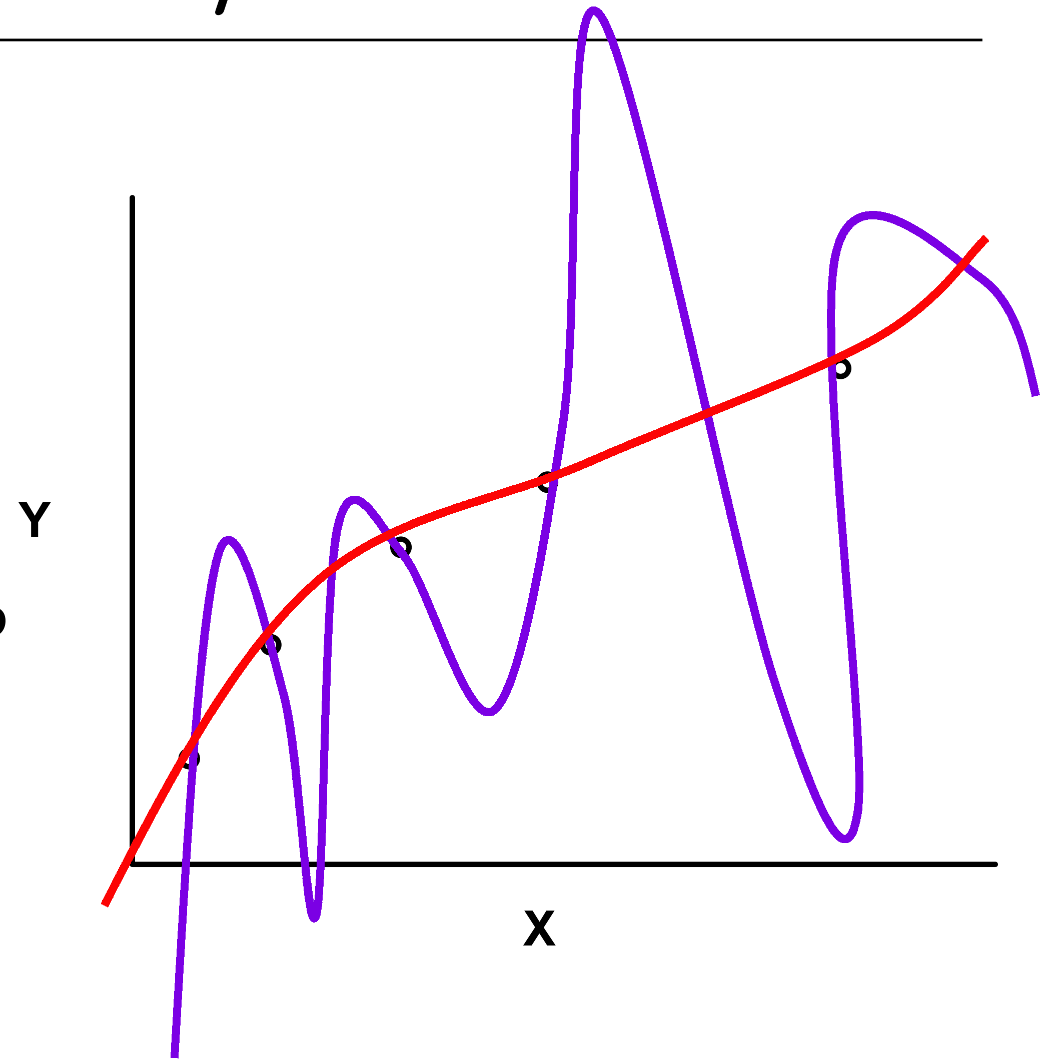


COMPENSATION

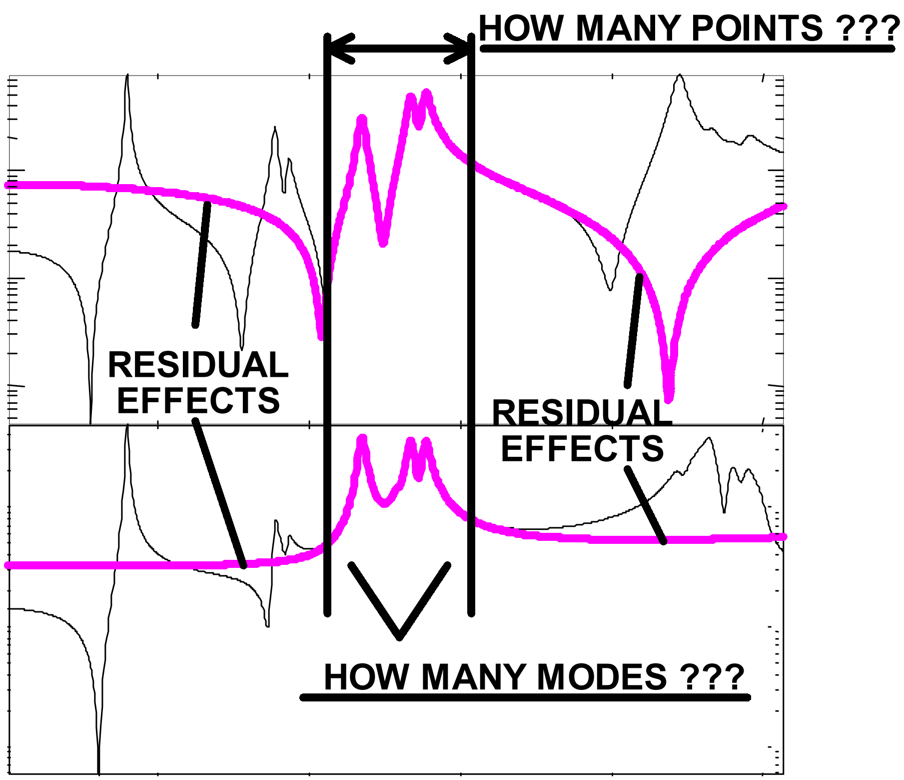
$$y = m\,x + b$$



WHICH DATA ???



Parameter Extraction Considerations



- *ORDER OF THE MODEL*
- *AMOUNT OF DATA TO BE USED*
- *COMPENSATION FOR RESIDUALS*

The test engineer identifies these items

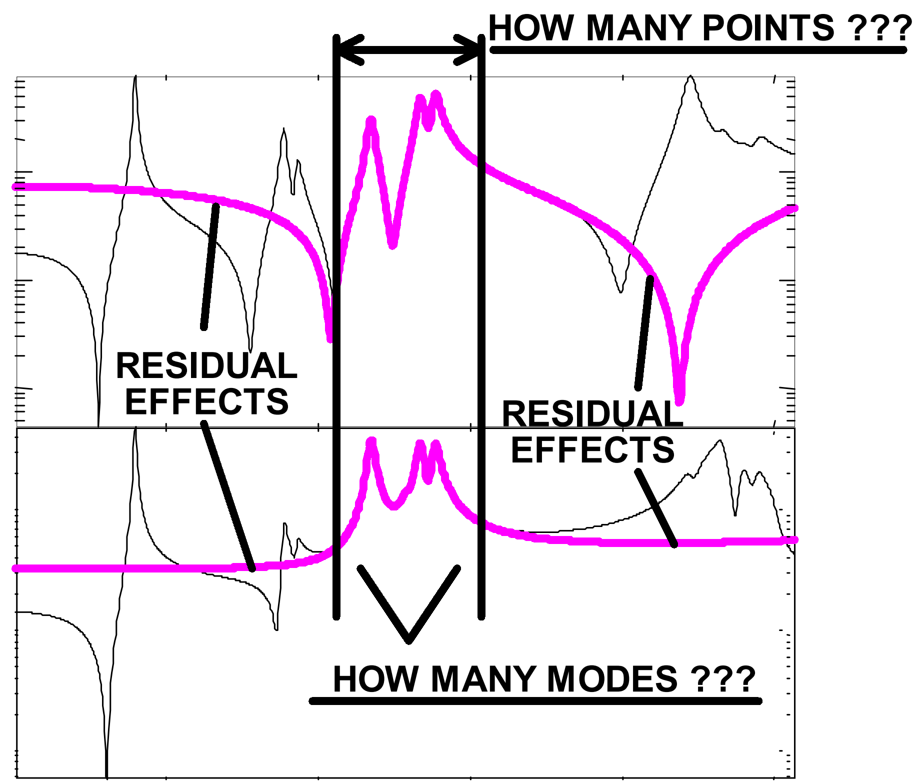
NOT THE SOFTWARE !!!

Parameter Extraction Considerations

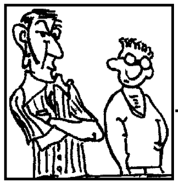
$$[H(s)] = \sum_{\text{terms}}^{\text{lower}} \frac{[A_k]}{(s - s_k)} + \frac{[A_k^*]}{(s - s_k^*)}$$

$$\sum_{k=i}^{j} \frac{[A_k]}{(s - s_k)} + \frac{[A_k^*]}{(s - s_k^*)}$$

$$\sum_{\text{terms}}^{\text{upper}} \frac{[A_k]}{(s - s_k)} + \frac{[A_k^*]}{(s - s_k^*)}$$

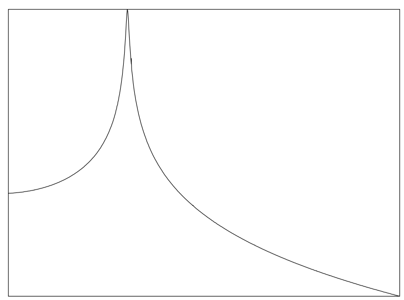


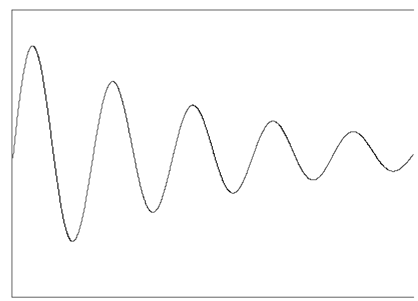
$$[H(s)] = \text{lower residuals} + \sum_{k=i}^{j} \frac{[A_k]}{(s-s_k)} + \frac{[A_k^*]}{(s-s_k^*)} + \text{upper}$$



Parameter Extraction Considerations

The basic equations can be cast in either the time or frequency domain





$$h(s) = \frac{a_1}{(s - p_1)} + \frac{a_1^*}{(s - p_1^*)}$$

$$h(t) = \frac{1}{m\omega_d} e^{-\sigma t} \sin \omega_d t$$

Parameter Extraction Considerations

MODAL PARAMETER ESTIMATION MODELS

Time representation

$$h_{ij\,(n)}(t) + a_1 h_{ij\,(n-1)}(t) + \cdots + a_{2n} h_{ij\,(n-2N)}(t) = 0$$

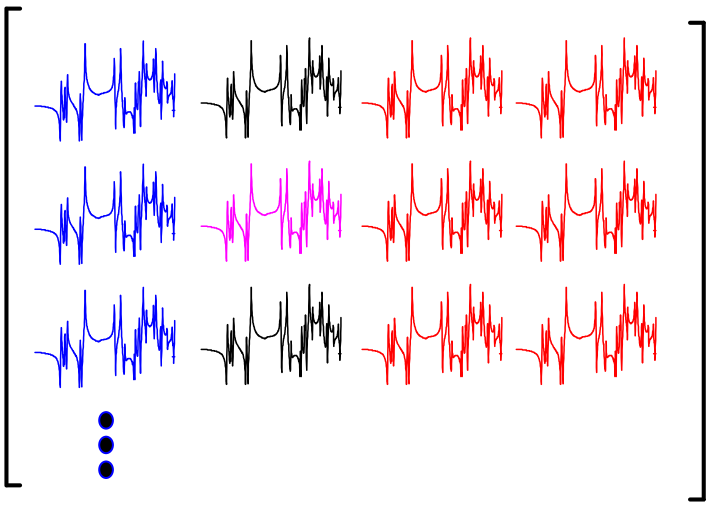
Frequency representation

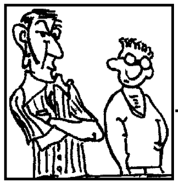
$$\left[(j\omega)^{2N} + a_1 (j\omega)^{2N-1} + \cdots + a_{2N}\right] h_{ij}(j\omega) =$$
$$\left[(j\omega)^{2M} + b_1 (j\omega)^{2M-1} + \cdots + b_{2M}\right]$$

Parameter Extraction Considerations

The FRF matrix contains redundant information regarding the system frequency, damping and mode shapes

Multiple referenced data can be used to obtain better estimates of modal parameters

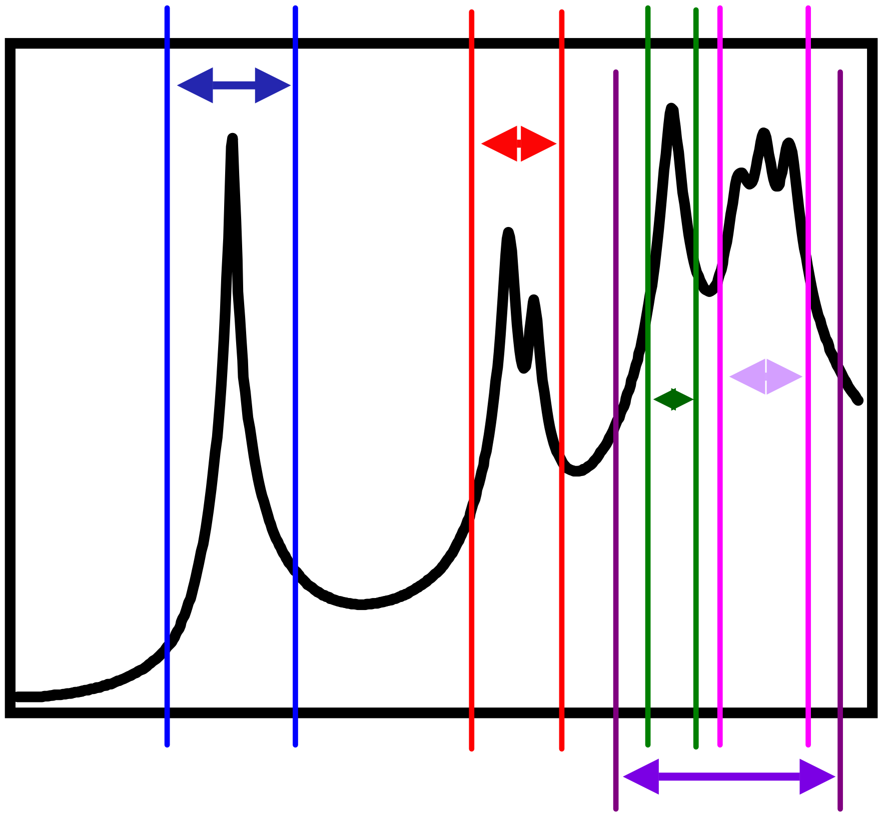




Selection of Bands

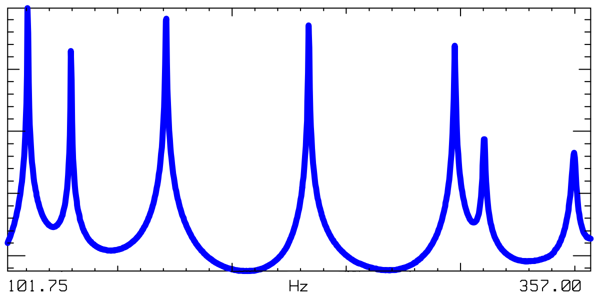
Select bands for possible SDOF or MDOF extraction for frequency domain technique.

Residuals ???   Complex ???

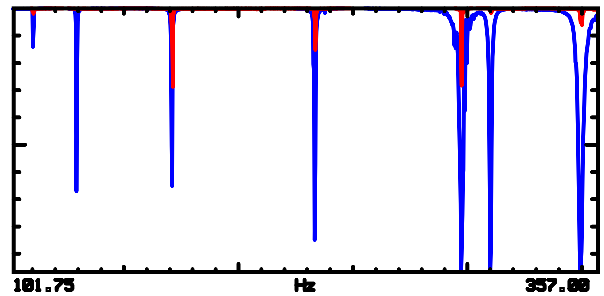


Mode Determination Tools

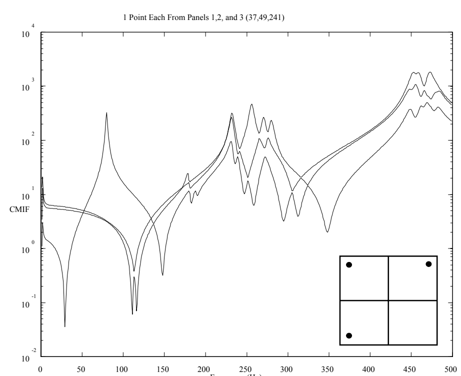
Summation



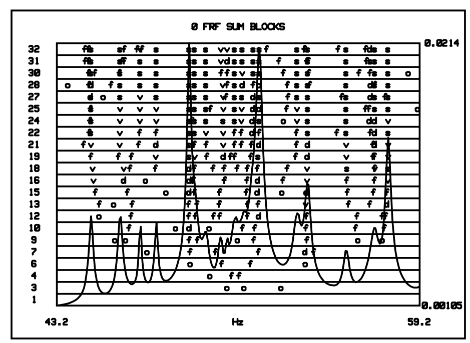
MIF



A variety of tools assist in the determination and selection of modes in the structure



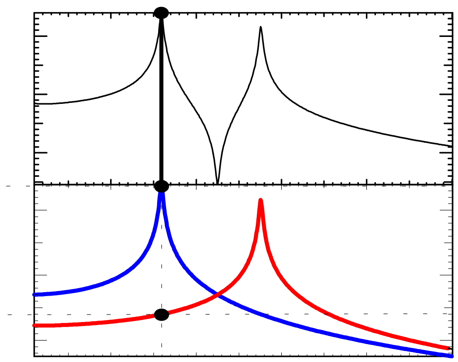
CMIF



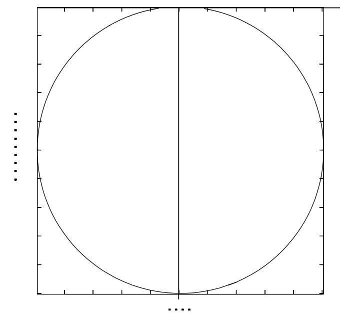
Stability Diagram

Modal Extraction Methods

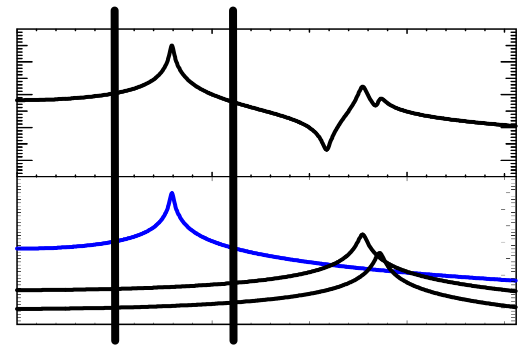
Peak Picking



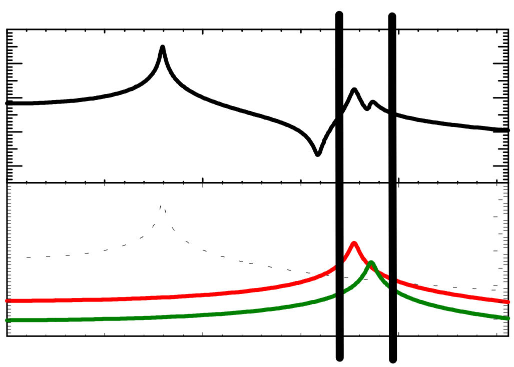
Circle Fitting



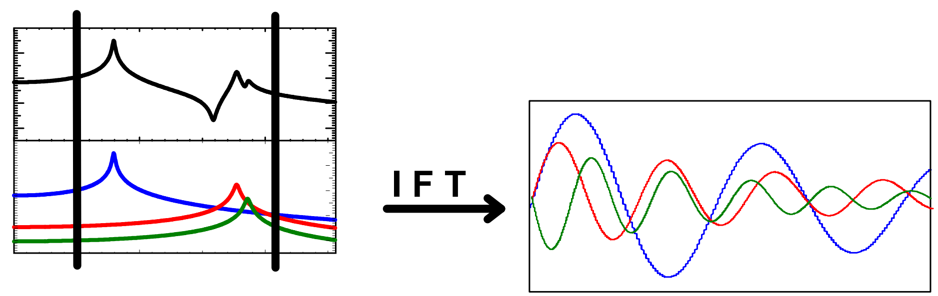
SDOF Polynomial



A multitude of techniques exist



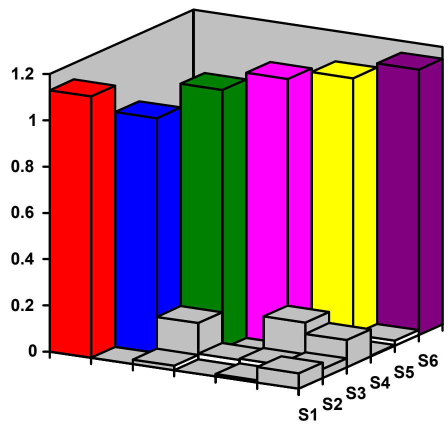
MDOF Polynomial Methods



Complex Exponential

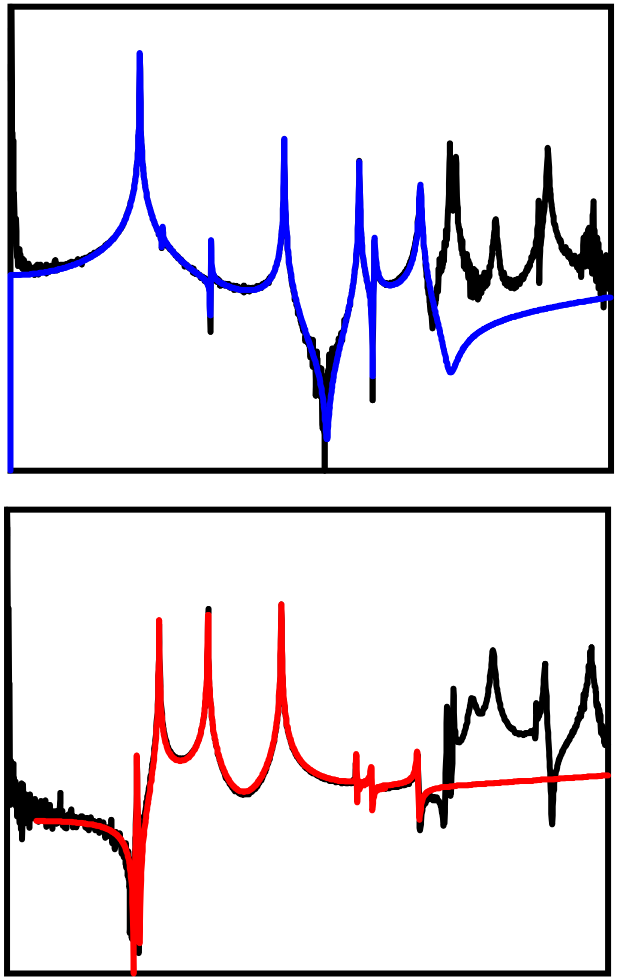
Model Validation

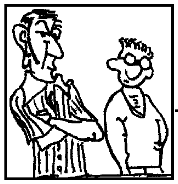
Validation tools exist to assure that an accurate model has been extracted from measured data



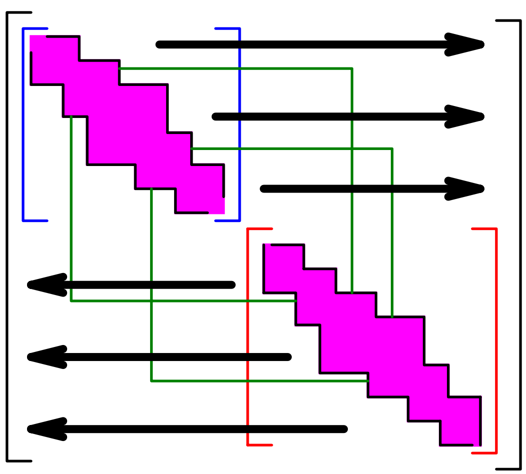
MAC

Synthesis





Linear Algebra Concepts



$$[A]=\left[\{u_1\}\quad\{u_2\}\quad\{u_3\}\quad\cdots\right]\begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} \{v_1\}^T \\ \{v_2\}^T \\ \{v_3\}^T \\ \vdots \end{bmatrix}$$

$$[U]=\begin{bmatrix} x & . & . & . & x \\ 0 & x & . & . & . \\ . & 0 & x & . & . \\ . & . & 0 & x & . \\ 0 & . & . & 0 & x \end{bmatrix}$$

$$[A]^{-1}=\frac{[\text{Adjoint}[A]]}{\text{Det}[A]}$$

$$[A]_{nm}\{X\}_m=\{B\}_n$$

$$[A]_{nm}=[V]_{nn}[S]_{nm}[U]^T_{mm}$$

$$\{X\}_m=[A]^g_{nm}\{B\}_n=\left[[V]_{nn}[S]_{nm}[U]^T_{mm}\right]^g\{B\}_n$$

$$\{X\}_m=\left[[U]_{mm}[S]^g_{nm}[V]^T_{nn}\right]\{B\}_n$$

Linear Algebra

The analytical treatment of structural dynamic systems naturally results in algebraic equations that are best suited to be represented through the use of matrices

Some common matrix representations and linear algebra concepts are presented in this section

Linear Algebra

Common analytical and experimental equations needing linear algebra techniques

$$[G_{yf}] = [H][G_{ff}] \quad \Rightarrow \quad [H] = [G_{yf}][G_{ff}]^{-1}$$

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\} \quad \Rightarrow \quad [[K]-\lambda[M]]\{x\}=0$$

$$[B(s)]\{x(s)\} = \{F(s)\} \quad \Rightarrow \quad [B(s)]^{-1} = [H(s)] = \frac{\mathrm{Adj}[B(s)]}{\det[B(s)]}$$

or

$$[H(s)] = [U]\begin{bmatrix} \ddots & & \\ & S & \\ & & \ddots \end{bmatrix}[L]^T$$

Matrix Notation

A matrix [A] can be described using row,column as

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}$$

(row , column)

$[A]^T$ -Transpose - interchange rows & columns

$[A]^H$ - Hermitian - conjugate transpose

Matrix Notation

A matrix [A] can have some special forms

Square

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Diagonal

$$[A]=\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & a_{44} & \\ & & & & a_{55} \end{bmatrix}$$

Triangular

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

Symmetric

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}$$

Toeplitz

$$[A]=\begin{bmatrix} a_{5} & a_{6} & a_{7} & a_{8} & a_{9} \\ a_{4} & a_{5} & a_{6} & a_{7} & a_{8} \\ a_{3} & a_{4} & a_{5} & a_{6} & a_{7} \\ a_{2} & a_{3} & a_{4} & a_{5} & a_{6} \\ a_{1} & a_{2} & a_{3} & a_{4} & a_{5} \end{bmatrix}$$

Vandermonde

$$[A]=\begin{bmatrix} 1 & a_{1} & a_{1}^{2} \\ 1 & a_{2} & a_{2}^{2} \\ 1 & a_{3} & a_{3}^{2} \\ 1 & a_{4} & a_{4}^{2} \end{bmatrix}$$

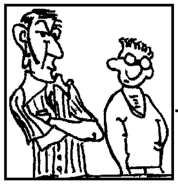
Matrix Manipulation

A matrix [C] can be computed from [A] & [B] as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} + a_{25}b_{51}$$

$$c_{ij} = \sum_{k} a_{ik} b_{kj}$$



Simple Set of Equations

A common form of a set of equations is

$$[A]\{x\} = [b]$$

Underdetermined ***# rows < # columns***

more unknowns than equations
(optimization solution)

Determined ***# rows = # columns***

equal number of rows and columns

Overdetermined ***# rows > # columns***

more equations than unknowns
(least squares or generalized inverse solution)

Simple Set of Equations

This set of equations has a unique solution

$$\begin{aligned} 2x - y &= 1 \\ -x + 2y - 1z &= 2 \\ -y + z &= 3 \end{aligned} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

whereas this set of equations does not

$$\begin{aligned} 2x - y &= 1 \\ -x + 2y - 1z &= 2 \\ 4x - 2y &= 2 \end{aligned} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 4 & -2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$$

Static Decomposition

A matrix [A] can be decomposed and written as

$$[A] = [L][U]$$

Where [L] and [U] are the lower and upper diagonal matrices that make up the matrix [A]

$$[L] = \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ x & x & x & 0 & 0 \\ x & x & x & x & 0 \\ x & x & x & x & x \end{bmatrix} \qquad [U] = \begin{bmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix}$$

Static Decomposition

Once the matrix [A] is written in this form then the solution for {x} can easily be obtained as

$$[A]=[L][U]$$

$$[U]\{X\}=[L]^{-1}[B]$$

Applications for static decomposition and inverse of a matrix are plentiful. Common methods are

Gaussian elimination   ***Crout reduction***

Gauss-Doolittle reduction   ***Cholesky reduction***

Eigenvalue Problems

Many problems require that two matrices [A] & [B] need to be reduced

$$[A]\{\ddot{x}\} + [B]\{x\} = \{Q(t)\} \quad \Rightarrow \quad \big[[B] - \lambda[A]\big]\{x\} = 0$$

Applications for solution of eigenproblems are plentiful. Common methods are

Jacobi ***Givens*** ***Householder***

Subspace Iteration ***Lanczos***

Singular Valued Decomposition

Any matrix can be decomposed using SVD

$$[A] = [U][S][V]^T$$

[U] - matrix containing left hand eigenvectors

[S] - diagonal matrix of singular values

[V] - matrix containing right hand eigenvectors

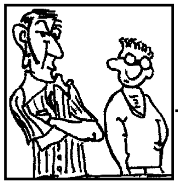
Singular Valued Decomposition

SVD allows this equation to be written as

$$[A]=\begin{bmatrix}\{u_1\} & \{u_2\} & \{u_3\} & \cdots\end{bmatrix}\begin{bmatrix}s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots\end{bmatrix}\begin{bmatrix}\{v_1\}^T \\ \{v_2\}^T \\ \{v_3\}^T \\ \vdots\end{bmatrix}$$

which implies that the matrix [A] can be written in terms of linearly independent pieces which form the matrix [A]

$$[A]=\{u_1\}s_1\{v_1\}^T+\{u_2\}s_2\{v_2\}^T+\{u_3\}s_3\{v_3\}^T+\quad\cdots$$



Singular Valued Decomposition

Assume a vector and singular value to be

$$u_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} \qquad \text{and} \qquad s_1 = 1$$

Then the matrix $[A_1]$ can be formed to be

$$[A_1] = \{u_1\}\, s_1\, \{u_1\}^T = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} [1] \{1 \quad 2 \quad 3\} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The size of matrix $[A_1]$ is (3x3) but its rank is 1
There is only one linearly independent piece of information in the matrix

Singular Valued Decomposition

Consider another vector and singular value to be

$$u_2 = \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} \qquad \text{and} \qquad s_2 = 1$$

Then the matrix $[A_2]$ can be formed to be

$$[A_2] = \{u_2\}\, s_2\, \{u_2\}^T = \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} [1] \{1 \quad 1 \quad -1\} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

The size and rank are the same as previous case
Clearly the rows and columns
are linearly related

Singular Valued Decomposition

Now consider a general matrix $[A_3]$ to be

$$[A_3]=\begin{bmatrix}2 & 3 & 2\\ 3 & 5 & 5\\ 2 & 5 & 10\end{bmatrix}=[A_1]+[A_2]$$

The characteristics of this matrix are not obvious at first glance.

Singular valued decomposition can be used to determine the characteristics of this matrix

Singular Valued Decomposition

The SVD of matrix [A_3] is

$$[A]=\left[\begin{Bmatrix}1\\2\\3\end{Bmatrix}\quad\begin{Bmatrix}1\\1\\-1\end{Bmatrix}\quad\begin{Bmatrix}0\\0\\0\end{Bmatrix}\right]\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\begin{bmatrix}\{1 & 2 & 3\}\\ \{1 & 1 & -1\}\\ \{0 & 0 & 0\}\end{bmatrix}$$

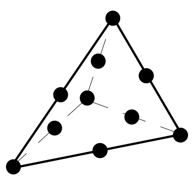
or

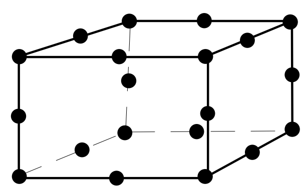
$$[A]=\begin{Bmatrix}1\\2\\3\end{Bmatrix}1\{1\quad 2\quad 3\}^{T}+\begin{Bmatrix}1\\1\\-1\end{Bmatrix}1\{1\quad 1\quad -1\}^{T}+\begin{Bmatrix}0\\0\\0\end{Bmatrix}0\{0\quad 0\quad 0\}^{T}$$

These are the independent quantities that make up the matrix which has a rank of 2

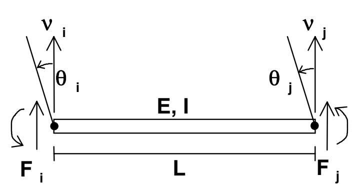
Linear Algebra Applications

The basic solid mechanics formulations as well as the individual elements used to generate a finite element model are described by matrices





$$[k]=\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L\end{bmatrix}$$

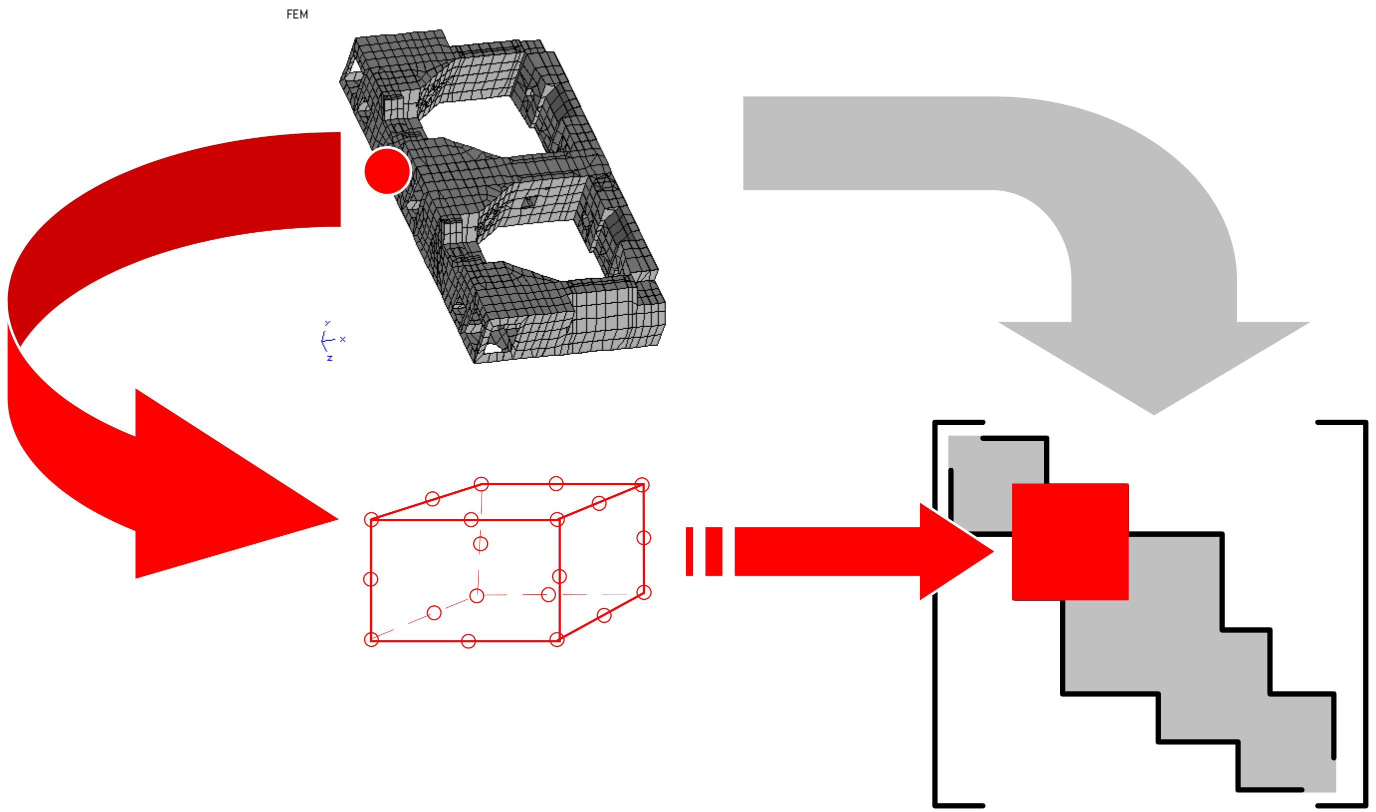


$$[m]=\frac{\rho AL}{420}\begin{bmatrix}156 & -22L & 54 & 13L\\ -22L & 4L^2 & -13L & -3L^2\\ 54 & -13L & 156 & 22L\\ 13L & -3L^2 & 22L & 4L^2\end{bmatrix}$$

$$\{\sigma\}=[C]\{\varepsilon\}\Rightarrow\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}$$

Linear Algebra Applications

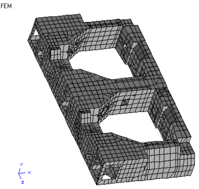
Finite element model development uses individual elements that are assembled into system matrices

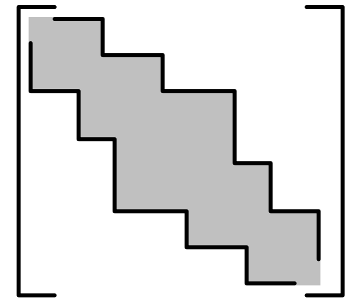


Linear Algebra Applications

Structural system equations - coupled

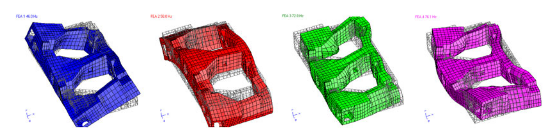
$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$





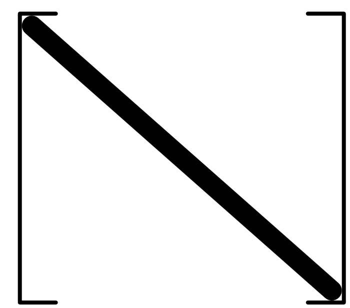
Eigensolution - eigenvalues & eigenvectors

$$[[K]-\lambda[M]]\{x\}=0$$



Modal space representation of equations - uncoupled

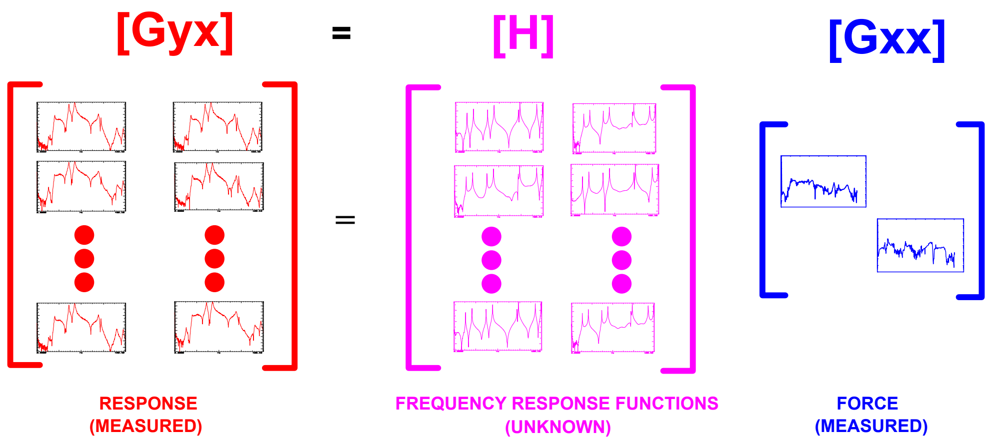
$$\begin{bmatrix} \ddots & & \\ & \overline{M} & \\ & & \ddots \end{bmatrix}\{\ddot{p}\}+\begin{bmatrix} \ddots & & \\ & \overline{C} & \\ & & \ddots \end{bmatrix}\{\dot{p}\}+\begin{bmatrix} \ddots & & \\ & \overline{K} & \\ & & \ddots \end{bmatrix}\{p\}=[U]^T\{F\}$$



Linear Algebra Applications

Multiple Input Multiple Output Data Reduction

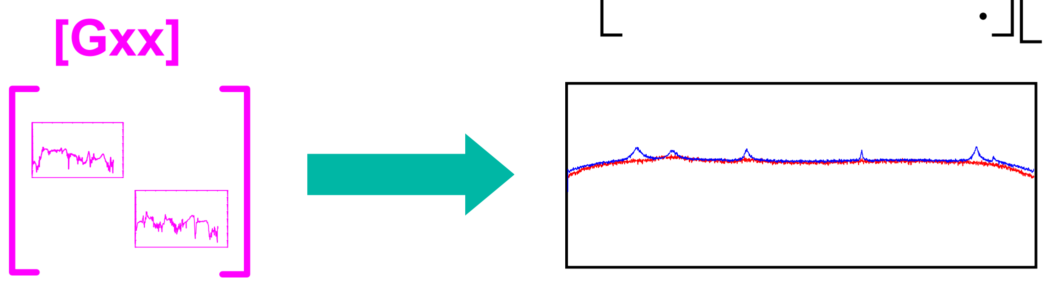
$$[G_{yx}] = [H][G_{xx}] \quad \Longrightarrow \quad [H] = [G_{yx}][G_{xx}]^{-1}$$



Matrix inversion can only be performed if the matrix [Gxx] has linearly independent inputs

Principal Component Analysis using SVD

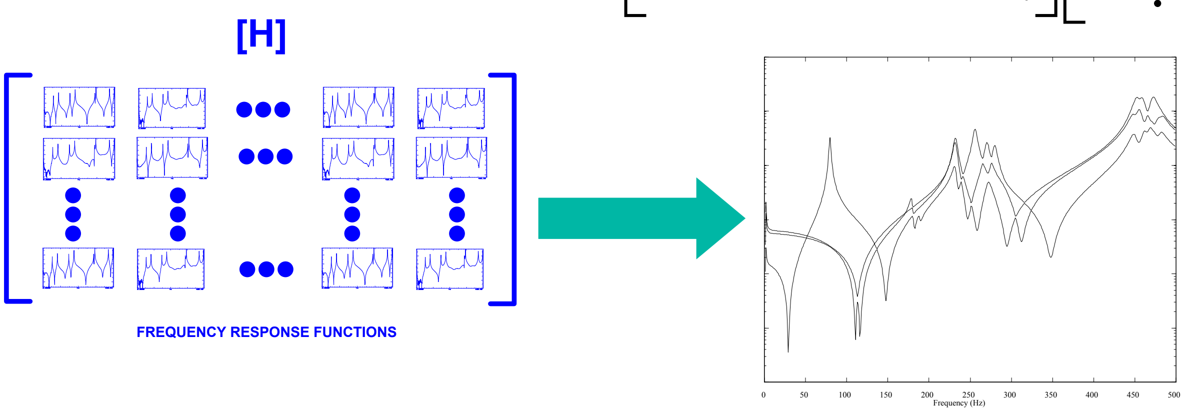
$$[G_{xx}] = \begin{bmatrix} \{u_1\} & \{u_2\} & \{0\} & \cdots \end{bmatrix} \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & 0 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \{v_1\}^T \\ \{v_2\}^T \\ \{0\}^T \\ \vdots \end{bmatrix}$$



SVD of the input excitation matrix identifies the rank of the matrix - that is an indication of how many linearly independent inputs exist

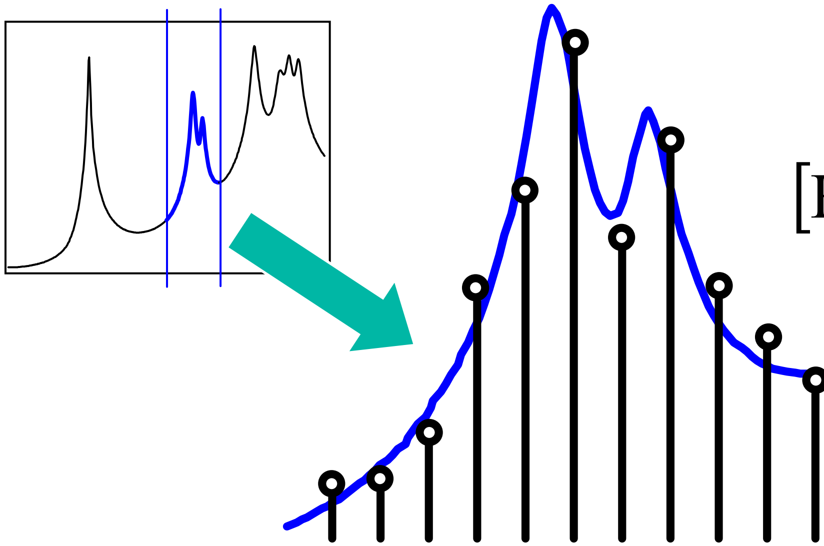
SVD of Multiple Reference FRF Data

$$[H]=\left[\{u_1\}\quad\{u_2\}\quad\{u_3\}\quad\cdots\right]\begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} \{v_1\}^T \\ \{v_2\}^T \\ \{v_3\}^T \\ \vdots \end{bmatrix}$$



SVD of the [H] matrix gives an indication of how many modes exist in the data

Least Squares or Generalized Inverse for Modal Parameter Estimation Techniques



$$[H(s)] = \sum_{k=i}^{j} \frac{[A_k]}{(s - s_k)} + \frac{\lfloor A_k^* \rfloor}{(s - s_k^*)}$$

Least squares error minimization of measured data to an analytical function

Linear Algebra Applications

Extended analysis and evaluation of systems

$$[K][U]=[M_I][U]\left[\lceil\omega^2\rfloor\right]$$

$$[U^T][K_I][U]=[U]^T[M_I][U]\left[\lceil\omega^2\rfloor\right]$$

$$[K_I]=[K_S]+[V]^T\left[\lceil\omega^2\rfloor+\overline{K}_S\right][V]$$
$$-\left[[K_S][U][U]^T[M_I]\right]-\left[[K_S][U][U]^T[M_I]\right]^T$$

$$[K_I]=[K_S]+[V]^T\left[\lceil\omega^2\rfloor+\overline{K}_S\right][V]-[[K_S][U][V]]-[[K_S][U][V]]^T$$

generally require matrix manipulation of some type

Linear Algebra Applications

Many other applications exist

Correlation

Model Updating

Advanced Data Manipulation

Operating Data

Rotating Equipment

Nonlinearities

Modal Parameter Estmation

and the list goes on and on