Presentation Topics

Intent
Modal Overview
SDOF Theory
MDOF Theory
Measurement Definitions
Excitation Considerations
MPE Concepts
Linear Algebra

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IMAC 19
Young Engineer Program

TUTORIAL:
Basics of Modal Analysis
Intent of Young Engineer Program

The intent of the Young Engineer Program is to expose the new or young engineer to some of the basic concepts and ideas concerning analytical and experimental modal analysis.

It is NOT intended to be a detailed treatment of this material.

Rather it is intended to prepare one for some of the in-depth papers presented at IMAC so that the novice has some appreciation of the detailed material presented in these papers.

This presentation is intended to identify the basic methodology and techniques currently employed in this field and to expose one to the typical modal jargon used in the field.
Could you explain modal analysis and how it is used for solving dynamic problems?
Analytical Modal Analysis

Equation of motion
\[
[M_n]\dddot{x}_n + [C_n]\ddot{x}_n + [K_n]x_n = \{F_n(t)\}
\]

Eigensolution
\[
[[K_n] - \lambda[M_n]]x_n = \{0\}
\]
Could you explain modal analysis for me?

Simple time-frequency response relationship

increasing rate of oscillation

force

response

time

frequency
Could you explain modal analysis for me?

Sine Dwell to Obtain Mode Shape Characteristics
Just what are the measurements called FRFs?

A simple input-output problem

Magnitude  |  Real
---|---
Phase  |  Imaginary
Just what are the measurements called FRFs?

Response at point 3 due to an input at point 3

Drive Point FRF

\[ h_{33} \]

\[ h_{32} \]

\[ h_{31} \]
Why is only one row/column of FRFs needed?

The third row of the FRF matrix - mode 1

The peak amplitude of the imaginary part of the FRF is a simple method to determine the mode shape of the system.
Why is only one row/column of FRFs needed?

The second row of the FRF matrix is similar.

The peak amplitude of the imaginary part of the FRF is a simple method to determine the mode shape of the system.
Why is only one row/column of FRFs needed?

Any row or column can be used to extract mode shapes

- as long as it is not the node of a mode!
More measurements better defines the shape
What's the difference between shaker and impact?

Theoretically -- -- -- NOTHING !!!
What measurements do I actually make?

Actual time signals
Analog anti-alias filter
Digitized time signals
Windowed time signals
Compute FFT of signal

Average auto/cross spectra
Compute FRF and Coherence
What's most important in impact testing?

Hammers and Tips

![Graphs showing FRF, coherence, and input power spectrum with frequency range from 0Hz to 800Hz and 0Hz to 200Hz.](image)
What’s most important in impact testing?

Leakage and Windows

ACTUAL TIME SIGNAL

SAMPLED SIGNAL

WINDOW WEIGHTING

WINDOWED TIME SIGNAL
What’s most important in shaker testing?

Different excitation techniques are available for obtaining good measurements.

- Random with Hanning
- Burst Random
- Sine Chirp

Random with Hanning
Burst Random
Sine Chirp
How do I get mode shapes from the FRFs?
How do I get mode shapes from the FRFs?

The FRF is made up from each individual mode contribution which is determined from the frequency, damping, and residue.
How do I get mode shapes from the FRFs?

The task for the modal test engineer is to determine the parameters that make up the pieces of the frequency response function.
How do I get mode shapes from the FRFs?

Mathematical routines help to determine the basic parameters that make up the FRF.
What is operating data?

Why and How Do Structures Vibrate?

INPUT TIME FORCE

\[ f(t) \]

FFT

\[ f(j\omega) \]

INPUT SPECTRUM

\[ y(t) \]

IFT

\[ y(j\omega) \]

OUTPUT SPECTRUM
What is operating data?

If I make measurements on a structure at an operating frequency, sometimes I get some deformation shapes that look pretty funky.

Maybe they are just noise?

Is that possible???
What is operating data?

But if I make a measurement at an operating frequency and it's close to a mode, I can easily see what appears to be one of the modes.
What is operating data?

And if I make a measurement at an operating frequency and its close to another mode, I can easily see what appears to be one of the modes.
What is operating data?

I think I just answered my own question !!!

I think I’m starting to understand this now !!!
**What is operating data?**

The modes of the structure act like filters which amplify and attenuate input excitations on a frequency basis.

![Diagram showing input and output spectra](image)
So what good is modal analysis?

The dynamic model can be used for studies to determine the effect of structural changes of the mass, damping and stiffness.
So what good is modal analysis?

Simulation, Prediction, Correlation, ... to name a few
Single Degree of Freedom Overview

\[
x(t) \quad f(t)
\]

\[
\begin{align*}
x(t) &= m \ddot{x}(t) + c \dot{x}(t) + kx(t) \\
X(t) &= m \frac{1}{\omega_n^2} (s^2 + 2\zeta\omega_n s + \omega_n^2) \\
h(s) &= \frac{1}{ms^2 + cs + k}
\end{align*}
\]
SDOF Definitions

Assumptions

• lumped mass
• stiffness proportional to displacement
• damping proportional to velocity
• linear time invariant
• 2nd order differential equations

\[ x(t) \]

\[ m \quad k \quad c \]
SDOF Equations

Equation of Motion

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t) \quad \text{or} \quad m \ddot{x} + c \dot{x} + kx = f(t) \]

Characteristic Equation

\[ ms^2 + cs + k = 0 \]

Roots or poles of the characteristic equation

\[ s_{1,2} = - \frac{c}{2m} \pm \sqrt{\left( \frac{c}{2m} \right)^2 + \frac{k}{m}} \]
**SDOF Definitions**

**Poles expressed as**

\[ s_{1,2} = -\zeta \omega_n \pm \sqrt{(\zeta \omega_n)^2 - \omega_n^2} = -\sigma \pm j\omega_d \]

**Damping Factor**

\[ \sigma = \zeta \omega_n \]

**Natural Frequency**

\[ \omega_n = \sqrt{\frac{k}{m}} \]

**% Critical Damping**

\[ \zeta = \frac{c}{c_c} \]

**Critical Damping**

\[ c_c = 2m\omega_n \]

**Damped Natural Frequency**

\[ \omega_d = \omega_n \sqrt{1-\zeta^2} \]
SDOF - Harmonic Excitation

\[
\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left(1 - \beta^2\right)^2 + (2\zeta\beta)^2}}
\]

\[
\phi = \tan^{-1}\left(\frac{2\zeta\beta}{1 - \beta^2}\right)
\]
SDOF - Damping Approximations

\[ Q = \frac{1}{2\zeta} = \frac{\omega_n}{\omega_2 - \omega_1} \]

\[ \delta = \ln \frac{x_1}{x_2} \approx 2\pi\zeta \]
**SDOF - Laplace Domain**

**Equation of Motion in Laplace Domain**

\[(ms^2 + cs + k)x(s) = f(s) \quad \text{with} \quad b(s) = (ms^2 + cs + k)\]

**System Characteristic Equation**

\[b(s) \cdot x(s) = f(s) \quad \text{and} \quad x(s) = b^{-1}(s)f(s) = h(s)f(s)\]

**System Transfer Function**

\[h(s) = \frac{1}{ms^2 + cs + k}\]
SDOF - Transfer Function

**Polynomial Form**

\[ h(s) = \frac{1}{ms^2 + cs + k} \]

**Pole-Zero Form**

\[ h(s) = \frac{1/m}{(s - p_1)(s - p_1^*)} \]

**Partial Fraction Form**

\[ h(s) = \frac{a_1}{(s - p_1)} + \frac{a_1^*}{(s - p_1^*)} \]

**Exponential Form**

\[ h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega t} \sin \omega_d t \]
**Residue**

\[
a_1 = \left. \frac{h(s)(s - p_1)}{s - p_1} \right|_{s \to p_1} = \frac{1}{2j \omega_d}
\]

*related to mode shapes*
SDOF - Frequency Response Function

\[ h(j\omega) = h(s) \mid_{s=j\omega} = \frac{a_1}{(j\omega - p_1)} + \frac{a_1^*}{(j\omega - p_1^*)} \]
**SDOF - Frequency Response Function**

**Bode Plot**

**Coincident-Quadrature Plot**

**Nyquist Plot**
## SDOF - Frequency Response Function

<table>
<thead>
<tr>
<th>DYNAMIC COMPLIANCE</th>
<th>DISPLACEMENT / FORCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOBILITY</td>
<td>VELOCITY / FORCE</td>
</tr>
<tr>
<td>INERTANCE</td>
<td>ACCELERATION / FORCE</td>
</tr>
<tr>
<td>DYNAMIC STIFFNESS</td>
<td>FORCE / DISPLACEMENT</td>
</tr>
<tr>
<td>MECHANICAL IMPEDANCE</td>
<td>FORCE / VELOCITY</td>
</tr>
<tr>
<td>DYNAMIC MASS</td>
<td>FORCE / ACCELERATION</td>
</tr>
</tbody>
</table>
Multiple Degree of Freedom Overview

\[
[B(s)]^{-1} = \frac{\text{adj}[B(s)]}{\det[B(s)]} = \frac{[A(s)]}{\det[B(s)]}
\]

\[
\begin{bmatrix}
\mathbf{M} & \mathbf{T} & \mathbf{K}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{p}}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{C}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\dot{p}}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{K}
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}
\end{bmatrix} = \begin{bmatrix}
\mathbf{U}^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}
\end{bmatrix}
\]
MDOF Definitions

Assumptions

• lumped mass
• stiffness proportional to displacement
• damping proportional to velocity
• linear time invariant
• 2nd order differential equations
MDOF Equations

Equation of Motion - Force Balance

\[
m_1 \ddot{x}_1 + (c_1 + c_2) \ddot{x}_1 - c_2 \ddot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t) \\
m_2 \ddot{x}_2 - c_2 \ddot{x}_1 + c_2 \ddot{x}_2 - k_2x_1 + k_2x_2 = f_2(t)
\]

Matrix Formulation

\[
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
(c_1 + c_2) & -c_2 \\
-c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
(k_1 + k_2) & -k_2 \\
-k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
f_1(t) \\
f_2(t)
\end{bmatrix}
\]

Matrices and Linear Algebra are important !!!
MDOF Equations

Equation of Motion
\[ [M]\ddot{x}+[C]\dot{x}+[K]x=F(t) \]

Eigensolution
\[ ([K]-\lambda[M])x=0 \]

Frequencies (eigenvalues) and Mode Shapes (eigenvectors)
\[ \begin{bmatrix} \Omega^2 \\ \end{bmatrix} = \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \end{bmatrix} \quad \text{and} \quad [U] = [\{u_1\} \ {u_2} \ \cdots] \]
**Modal Space Transformation**

**Modal transformation**

\[ \{x\} = [U]\{p\} = \{u_1\ \ u_2\ \ \ldots\ \ p_2\ \ \vdots\ \} \]

**Projection operation**


**Modal equations (uncoupled)**

\[
\begin{bmatrix}
\bar{m}_1 \\
\bar{m}_2 \\
\vdots
\end{bmatrix} \begin{bmatrix}
\ddot{p}_1 \\
\ddot{p}_2 \\
\vdots
\end{bmatrix} + \begin{bmatrix}
\ddot{c}_1 \\
\ddot{c}_2 \\
\vdots
\end{bmatrix} \begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\vdots
\end{bmatrix} + \begin{bmatrix}
\bar{k}_1 \\
\bar{k}_2 \\
\vdots
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\{u_1\}^T\{F\} \\
\{u_2\}^T\{F\} \\
\vdots
\end{bmatrix}
\]
Modal Space Transformation

Diagonal Matrices -

\[
\begin{bmatrix}
\mathbf{M}
\end{bmatrix}\{\ddot{\mathbf{p}}\} + \begin{bmatrix}
\mathbf{C}
\end{bmatrix}\{\dot{\mathbf{p}}\} + \begin{bmatrix}
\mathbf{K}
\end{bmatrix}\{\mathbf{p}\} = [\mathbf{U}]^T\{\mathbf{F}\}
\]

Highly coupled system

transformed into

simple system
Modal Space Transformation

\[ [M] \ddot{\{\mathbf{x}\}} + [C] \dot{\{\mathbf{x}\}} + [K] \{\mathbf{x}\} = \{\mathbf{F}(t)\} \]

\[ \{\mathbf{x}\} = [U]\{\mathbf{p}\} = \{u_1\}p_1 + \{u_2\}p_2 + \{u_3\}p_3 \]

MODAL \rightarrow SPACE

\[ [\tilde{M}] \ddot{\{\mathbf{p}\}} + [\tilde{C}] \dot{\{\mathbf{p}\}} + [\tilde{K}] \{\mathbf{p}\} = [U]^T \{\mathbf{F}(t)\} \]

\[ \{u_1\}p_1 + \{u_2\}p_2 + \{u_3\}p_3 \]
**MDOF - Laplace Domain**

**Laplace Domain Equation of Motion**

\[
[M]s^2 + [C]s + [K] \{x(s)\} = 0 \quad \Rightarrow \quad [B(s)]\{x(s)\} = 0
\]

**System Characteristic (Homogeneous) Equation**

\[
[M]s^2 + [C]s + [K] = 0 \quad \Rightarrow \quad p_k = -\sigma_k \pm j\omega_{dk}
\]

Damping  \quad Frequency
MDOF - Transfer Function

System Equation

\[ [B(s)]\{x(s)\} = \{F(s)\} \Rightarrow [H(s)] = [B(s)]^{-1} = \frac{\{x(s)\}}{\{F(s)\}} \]

System Transfer Function

\[ [B(s)]^{-1} = [H(s)] = \frac{\text{Adj}[B(s)]}{\det[B(s)]} = \frac{[A(s)]}{\det[B(s)]]} \]

- \([A(s)]\) Residue Matrix → Mode Shapes
- \(\det[B(s)]\) Characteristic Equation → Poles
**MDOF - Residue Matrix and Mode Shapes**

Transfer Function evaluated at one pole

\[
[H(s)]_{s=s_k} = \{u_k\} \frac{q_k}{s-p_k} \{u_k\}^T
\]

can be expanded for all modes

\[
[H(s)] = \sum_{k=1}^{m} \frac{q_k \{u_k\} \{u_k\}^T}{(s-p_k)} + \frac{q_k \{u^*_k\} \{u^*_k\}^T}{(s-p^*_k)}
\]
MDOF - Residue Matrix and Mode Shapes

Residues are related to mode shapes as

\[
[A(s)]_k = q_k \{u_k\}\{u_k\}^T
\]

\[
\begin{bmatrix}
a_{11k} & a_{12k} & a_{13k} & \cdots \\
a_{21k} & a_{22k} & a_{23k} & \cdots \\
a_{31k} & a_{32k} & a_{33k} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
= q_k
\begin{bmatrix}
u_{1k} & u_{1k} & u_{1k} & \cdots \\
u_{2k} & u_{2k} & u_{2k} & \cdots \\
u_{3k} & u_{3k} & u_{3k} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
MDOF - Drive Point FRF

\[ h_{ij}(j\omega) = \frac{a_{ij}}{(j\omega - p_i)} + \frac{a_{ij}^*}{(j\omega - p_i^*)} + \frac{a_{ij2}}{(j\omega - p_2)} + \frac{a_{ij2}^*}{(j\omega - p_2^*)} + \frac{a_{ij3}}{(j\omega - p_3)} + \frac{a_{ij3}^*}{(j\omega - p_3^*)} \]

\[ h_{ij}(j\omega) = \frac{q_{ij1}u_{ij1}}{(j\omega - p_1)} + \frac{q_{ij1}u_{ij1}^*}{(j\omega - p_1^*)} + \frac{q_{ij2}u_{ij2}}{(j\omega - p_2)} + \frac{q_{ij2}u_{ij2}^*}{(j\omega - p_2^*)} + \frac{q_{ij3}u_{ij3}}{(j\omega - p_3)} + \frac{q_{ij3}u_{ij3}^*}{(j\omega - p_3^*)} \]
MDOF - FRF using Residues or Mode Shapes

\[ h_{ij}(j\omega) = \frac{a_{ij1}}{(j\omega - p_1)} + \frac{a_{ij1}^*}{(j\omega - p_1^*)} + \frac{a_{ij2}}{(j\omega - p_2)} + \frac{a_{ij2}^*}{(j\omega - p_2^*)} + \cdots \]

\[ h_{ij}(j\omega) = \frac{q_1 u_{i1} u_{j1}}{(j\omega - p_1)} + \frac{q_1 u_{i1}^* u_{j1}^*}{(j\omega - p_1^*)} + \frac{q_2 u_{i2} u_{j2}}{(j\omega - p_2)} + \frac{q_2 u_{i2}^* u_{j2}^*}{(j\omega - p_2^*)} + \cdots \]
Modal Analysis & Controls Laboratory

**MDOF Overview**

**Time / Frequency / Modal Representation**

**Physical** → **Time** → **Frequency** → **Analytical**

- **Mode 1**: 1 \( m \) \( k \) \( c_1 \) \( f_1p_1 \)
- **Mode 2**: 2 \( m \) \( k \) \( c_2 \) \( f_2p_2 \)
- **Mode 3**: 3 \( m \) \( k \) \( c_3 \) \( f_3p_3 \)

**Physical Mode**: Graphs showing displacement over time.

**Time Mode**: Graphs showing time-domain response.

**Frequency Mode**: Graphs showing frequency response.

**Analytical Mode**: Diagrams of mechanical systems representing each mode.
Overview Analytical and Experimental Modal Analysis

\[ [B(s)] = [M]s^2 + [C]s + [K] \]
\[ [B(s)]^{-1} = [H(s)] \]

\[ [U] \]

LAPLACE DOMAIN

TRANSFER FUNCTION

MODAL PARAMETER ESTIMATION

\[ H(j\omega) = \frac{X(j\omega)}{F_i(j\omega)} \]

FFT

FINITE ELEMENT MODEL

[\begin{bmatrix} A(s) \end{bmatrix}]

\[ \text{det} [B(s)] \]

[\begin{bmatrix} K - \lambda M \end{bmatrix} \{X\} = 0 \]

LARGE DOF MISMATCH

ANALYTICAL MODEL REDUCTION

MODAL TEST

EXPERIMENTAL MODAL MODEL EXPANSION

MDOF Overview
Measurement Definitions

\[ H(u(t), v(t)) = n(t) + m(t) \]

\[ x(t) + y(t) \]

ACTUAL

NOISE

MEASURED
Measurement Definitions

Actual time signals
Analog anti-alias filter
Digitized time signals
Windowed time signals
Compute FFT of signal
Average auto/cross spectra
Compute FRF and Coherence
Leakage

A periodic signal is represented by a signal that repeats itself at regular intervals. The frequency domain representation of a periodic signal is a single peak, indicating the frequency at which the signal repeats.

A non-periodic signal does not repeat itself at regular intervals. The frequency domain representation of a non-periodic signal shows multiple peaks, indicating the presence of multiple frequencies.

Leakage due to signal distortion occurs when the signal is not strictly periodic, leading to inaccuracies in the frequency domain analysis.
**Windows**

Time weighting functions are applied to **minimize** the effects of leakage

- Rectangular
- Hanning
- Flat Top
- and many others

**Windows DO NOT** eliminate leakage !!!
Special windows are used for impact testing

Force window

Exponential Window
**Measurements - Linear Spectra**

- **x(t)** - time domain input to the system
- **y(t)** - time domain output to the system
- **Sx(f)** - linear Fourier spectrum of **x(t)**
- **Sy(f)** - linear Fourier spectrum of **y(t)**
- **H(f)** - system transfer function
- **h(t)** - system impulse response

**Diagram:**
- INPUT: **x(t)**
- SYSTEM: **h(t)**
- OUTPUT: **y(t)**
- **Sx(f)** → **H(f)** → **Sy(f)**

**FFT & IFT**

**TIME**

**FREQUENCY**
Measurement Definitions - Linear Spectra

\[
x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} df
\]
\[
S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt
\]

\[
y(t) = \int_{-\infty}^{+\infty} S_y(f) e^{j2\pi ft} df
\]
\[
S_y(f) = \int_{-\infty}^{+\infty} y(t) e^{-j2\pi ft} dt
\]

\[
h(t) = \int_{-\infty}^{+\infty} H(f) e^{j2\pi ft} df
\]
\[
H(f) = \int_{-\infty}^{+\infty} h(t) e^{-j2\pi ft} dt
\]

Note: \( S_x \) and \( S_y \) are complex valued functions
### Measurements - Power Spectra

**Rxx(t)** - autocorrelation of the input signal $x(t)$

**Ryy(t)** - autocorrelation of the output signal $y(t)$

**Ryx(t)** - cross correlation of $y(t)$ and $x(t)$

**Gxx(f)** - autopower spectrum of $x(t)$

$$G_{xx}(f) = S_x(f) \cdot S_x^*(f)$$

**Gyy(f)** - autopower spectrum of $y(t)$

$$G_{yy}(f) = S_y(f) \cdot S_y^*(f)$$

**Gyx(f)** - cross power spectrum of $y(t)$ and $x(t)$

$$G_{yx}(f) = S_y(f) \cdot S_x^*(f)$$
Measurements - Linear Spectra

\[ R_{xx}(\tau) = \mathbb{E}[x(t), x(t + \tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} x(t)x(t + \tau)dt \]

\[ G_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j2\pi ft}d\tau = S_x(f)S^*_x(f) \]

\[ R_{yy}(\tau) = \mathbb{E}[y(t), y(t + \tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} y(t)y(t + \tau)dt \]

\[ G_{yy}(f) = \int_{-\infty}^{+\infty} R_{yy}(\tau)e^{-j2\pi ft}d\tau = S_y(f)S^*_y(f) \]

\[ R_{yx}(\tau) = \mathbb{E}[y(t), x(t + \tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} y(t)x(t + \tau)dt \]

\[ G_{yx}(f) = \int_{-\infty}^{+\infty} R_{yx}(\tau)e^{-j2\pi ft}d\tau = S_y(f)S^*_x(f) \]
Measurements - Derived Relationships

\[ S_y = H S_x \]

\( H_1 \) formulation
- susceptible to noise on the input
- underestimates the actual \( H \) of the system

\[ S_y \cdot S_x^* = H S_x \cdot S_x^* \]

\[ H = \frac{S_y \cdot S_x^*}{S_x \cdot S_x^*} = \frac{G_{yx}}{G_{xx}} \]

\( H_2 \) formulation
- susceptible to noise on the output
- overestimates the actual \( H \) of the system

\[ S_y \cdot S_y^* = H S_x \cdot S_y^* \]

\[ H = \frac{S_y \cdot S_y^*}{S_x \cdot S_y^*} = \frac{G_{yy}}{G_{xy}} \]

COHERENCE

\[ \gamma_{xy}^2 = \frac{(S_y \cdot S_x^*) (S_x \cdot S_y^*)}{(S_x \cdot S_x^*) (S_y \cdot S_y^*)} = \frac{G_{yx}}{G_{xx}} \cdot \frac{G_{yy}}{G_{xy}} = \frac{H_1}{H_2} \]
Measurements - Noise

\[
H = \frac{G_{uv}}{G_{uu}}
\]

\[
H_1 = H \left[ \frac{1}{1+ \frac{G_{nn}}{G_{uu}}} \right]
\]

\[
H_2 = H \left[ 1+ \frac{G_{mm}}{G_{vv}} \right]
\]
Measurements - Auto Power Spectrum

INPUT FORCE

\[ x(t) \]

OUTPUT RESPONSE

\[ G_{xx}(f) \]

AVERAGED INPUT

POWER SPECTRUM

AVERAGED OUTPUT

POWER SPECTRUM

\[ y(t) \]
Measurements - Cross Power Spectrum

\[ G_{xx}(f) \]
\[ G_{yy}(f) \]
\[ G_{yx}(f) \]
Measurements - Frequency Response Function

\[ G_{xx}(f) \quad G_{yx}(f) \quad G_{yy}(f) \]

\[ H(f) \]
Measurements - FRF & Coherence

COHERENCE

FREQUENCY RESPONSE FUNCTION
Excitation Considerations
Excitation Considerations - Impact

The force spectrum can be customized to some extent through the use of hammer tips with various hardmesses.
Excitation Considerations - Impact/Exponential

The excitation must be sufficient to excite all the modes of interest over the desired frequency range.
Excitation Considerations - Impact/Exponential

The response due to impact excitation may need an exponential window if leakage is a concern.
Excitation Considerations - Shaker Excitation

Leakage is a serious concern
Accurate FRFs are necessary

Special excitation techniques can be used which will result in leakage free measurements without the use of a window as well as other techniques.
Excitation Considerations - MIMO

Multiple referenced FRFs are obtained from MIMO test

Energy is distributed better throughout the structure making better measurements possible
Large or complicated structures require special attention.
Excitation Considerations - MIMO

Measurements are developed in a similar fashion to the single input single output case but using a matrix formulation.

\[
\begin{bmatrix}
G_{XF} & \vdots \\
\vdots & H \\
\vdots & \vdots \\
G_{FF}
\end{bmatrix}
\]

\[
[H] = \begin{bmatrix}
H_{11} & H_{12} & \cdots & H_{1,Ni} \\
H_{21} & H_{22} & \cdots & H_{2,Ni} \\
\vdots & \vdots & \ddots & \vdots \\
H_{No,1} & H_{No,2} & \cdots & H_{No,Ni}
\end{bmatrix}
\]

where

\[
[H][G_{XF}][G_{FF}]^{-1}
\]

No - number of outputs
Ni - number of inputs
Excitation Considerations - MIMO

Measurements on the same structure can show tremendously different modal densities depending on the location of the measurement.

Source: Michigan Technological University Dynamic Systems Laboratory
Modal Parameter Estimation Concepts

\[ [H(s)] = \sum_{\text{lower terms}} \frac{[A_k]}{(s-s_k)} + \frac{[A^*_k]}{(s-s_k^*)} + \sum_{k=i}^{j} \frac{[A_k]}{(s-s_k)} + \frac{[A^*_k]}{(s-s_k^*)} + \sum_{\text{upper terms}} \frac{[A_k]}{(s-s_k)} + \frac{[A^*_k]}{(s-s_k^*)} \]
Parameter Estimation Concepts

NO COMPENSATION

\[ y = mx \]

COMPENSATION

\[ y = mx + b \]

WHICH DATA ???
Parameter Extraction Considerations

The test engineer identifies these items

NOT THE SOFTWARE !!!

- Order of the model
- Amount of data to be used
- Compensation for residuals
- How many points
- How many modes
- Residual effects

Modal Parameter Estimation Concepts
Dr. Peter Avitabile
Modal Analysis & Controls Laboratory
Parameter Extraction Considerations

\[
[H(s)] = \sum_{\text{lower terms}} \frac{[A_k]}{(s - s_k)} + \frac{[A^*_k]}{(s - s_k^*)}
\]

\[
= \sum_{k=i}^{j} \frac{[A_k]}{(s - s_k)} + \frac{[A^*_k]}{(s - s_k^*)}
\]

\[
= \sum_{\text{upper terms}} \frac{[A_k]}{(s - s_k)} + \frac{[A^*_k]}{(s - s_k^*)}
\]

\[
[H(s)] = \text{lower residuals} + \sum_{k=i}^{j} \frac{[A_k]}{(s - s_k)} + \frac{[A^*_k]}{(s - s_k^*)} + \text{upper}
\]
Parameter Extraction Considerations

The basic equations can be cast in either the time or frequency domain

\[ h(s) = \frac{a_1}{(s - p_1)} + \frac{a_1^*}{(s - p_1^*)} \]

\[ h(t) = \frac{1}{m\omega_d} e^{-\sigma t} \sin \omega_d t \]
Parameter Extraction Considerations

MODAL PARAMETER ESTIMATION MODELS

Time representation

\[ h_{ij(n)}(t) + a_1 h_{ij(n-1)}(t) + \cdots + a_{2n} h_{ij(n-2N)}(t) = 0 \]

Frequency representation

\[
\left[ (j\omega)^{2N} + a_1 (j\omega)^{2N-1} + \cdots + a_{2N} \right] h_{ij}(j\omega) = \\
\left[ (j\omega)^{2M} + b_1 (j\omega)^{2M-1} + \cdots + b_{2M} \right]
\]
Parameter Extraction Considerations

The FRF matrix contains redundant information regarding the system frequency, damping and mode shapes.

Multiple referenced data can be used to obtain better estimates of modal parameters.
Selection of Bands

Select bands for possible SDOF or MDOF extraction for frequency domain technique.

Residuals ??? Complex ???
Mode Determination Tools

**Summation**

**MIF**

A variety of tools assist in the determination and selection of modes in the structure

**CMIF**

**Stability Diagram**
Modal Extraction Methods

A multitude of techniques exist

- Peak Picking
- Circle Fitting
- SDOF Polynomial
- MDOF Polynomial Methods
- Complex Exponential
- IFT
Validation tools exist to assure that an accurate model has been extracted from measured data.
Linear Algebra Concepts

\[ [A]^{-1} = \text{Adjoint}[A] \]

\[ \text{Det}[A] \]

\[ [A] = [u_1, u_2, u_3, \ldots] \]

\[ [s_1, s_2, s_3, \ldots] \]

\[ [v_1^T, v_2^T, v_3^T, \ldots] \]

\[ [U] \]

\[ x_\ldots x \]

\[ 0 x \ldots \]

\[ \ldots 0 x \]

\[ 0 \ldots 0 x \]

\[ [A]_{nm}[X]_m = [B]_n \]

\[ [A]_{nm} = [V]_m [S]_{nm} [U]_m^T \]

\[ [X]_m = [A]_{nm} [B]_n = [V]_m [S]_{nm} [U]_m^T \]

\[ [X]_m = [U]_{nm} [S]_{nm} [V]_m^T \]
Linear Algebra

The analytical treatment of structural dynamic systems naturally results in algebraic equations that are best suited to be represented through the use of matrices

Some common matrix representations and linear algebra concepts are presented in this section
Linear Algebra

Common analytical and experimental equations needing linear algebra techniques

\[ [G_{yf}] = [H][G_{ff}] \]

\[ [H] = [G_{yf}]^{-1}[G_{ff}] \]

\[ [M][\ddot{x}]+[C][\dot{x}]+[K][x]=[F(t)] \]

\[ [(K)-\lambda[M]][x]=0 \]

\[ [B(s)][x(s)] = \{F(s)\} \]

\[ [B(s)]^{-1} = [H(s)] = \frac{\text{Adj}[B(s)]}{\text{det}[B(s)]} \]

Or

\[ [H(s)] = [U]\begin{bmatrix} \ddots & S & \ddots \end{bmatrix}[L]^T \]
**Matrix Notation**

A matrix $[A]$ can be described using row, column as

$$[A] = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44} \\
  a_{51} & a_{52} & a_{53} & a_{54}
\end{bmatrix}$$

(row, column)

$[A]^T$ - Transpose - interchange rows & columns

$[A]^H$ - Hermitian - conjugate transpose
Matrix Notation

A matrix \([A]\) can have some special forms

**Square**
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}
\]

**Diagonal**
\[
\begin{bmatrix}
a_{11} \quad & & & & \\
& a_{22} \quad & & & \\
& & a_{33} \quad & & \\
& & & a_{44} \quad & \\
& & & & a_{55}
\end{bmatrix}
\]

**Triangular**
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
0 & a_{22} & a_{23} & a_{24} & a_{25} \\
0 & 0 & a_{33} & a_{34} & a_{35} \\
0 & 0 & 0 & a_{44} & a_{45} \\
0 & 0 & 0 & 0 & a_{55}
\end{bmatrix}
\]

**Symmetric**
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\
a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\
a_{15} & a_{25} & a_{35} & a_{45} & a_{55}
\end{bmatrix}
\]

**Toeplitz**
\[
\begin{bmatrix}
a_5 & a_6 & a_7 & a_8 & a_9 \\
a_4 & a_5 & a_6 & a_7 & a_8 \\
a_3 & a_4 & a_5 & a_6 & a_7 \\
a_2 & a_3 & a_4 & a_5 & a_6 \\
a_1 & a_2 & a_3 & a_4 & a_5
\end{bmatrix}
\]

**Vandermonde**
\[
\begin{bmatrix}
1 & a_1 & a_1^2 \\
1 & a_2 & a_2^2 \\
1 & a_3 & a_3^2 \\
1 & a_4 & a_4^2
\end{bmatrix}
\]
Matrix Manipulation

A matrix \([C]\) can be computed from \([A]\) & \([B]\) as

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35}
\end{bmatrix}
\begin{bmatrix}
  b_{11} \\
  b_{21} \\
  b_{31} \\
  b_{41} \\
  b_{51}
\end{bmatrix}
= \begin{bmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22} \\
  c_{31} & c_{32}
\end{bmatrix}
\]

\[
c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} + a_{25}b_{51}
\]

\[
c_{ij} = \sum_{k} a_{ik}b_{kj}
\]
Simple Set of Equations

A common form of a set of equations is

$$[A] \{x\} = [b]$$

**Underdetermined**

\# rows < \# columns
more unknowns than equations
(optimization solution)

**Determined**

\# rows = \# columns
equal number of rows and columns

**Overdetermined**

\# rows > \# columns
more equations than unknowns
(least squares or generalized inverse solution)
Simple Set of Equations

This set of equations has a unique solution

\[
\begin{align*}
2x - y &= 1 \\
-x + 2y - 1z &= 2 \\
-y + z &= 3
\end{align*}
\]

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

whereas this set of equations does not

\[
\begin{align*}
2x - y &= 1 \\
-x + 2y - 1z &= 2 \\
4x - 2y &= 2
\end{align*}
\]

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
4 & -2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix}
\]
Static Decomposition

A matrix \([A]\) can be decomposed and written as

\[
[A] = [L][U]
\]

Where \([L]\) and \([U]\) are the lower and upper diagonal matrices that make up the matrix \([A]\)

\[
[L] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 \\
X & X & 0 & 0 & 0 \\
X & X & X & 0 & 0 \\
X & X & X & X & 0 \\
X & X & X & X & X
\end{bmatrix}
\]

\[
[U] = \begin{bmatrix}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & X
\end{bmatrix}
\]
Static Decomposition

Once the matrix $[A]$ is written in this form then the solution for $\{x\}$ can easily be obtained as

$$[A] = [L][U]$$

$$[U]\{X\} = [L]^{-1}[B]$$

Applications for static decomposition and inverse of a matrix are plentiful. Common methods are

- Gaussian elimination
- Crout reduction
- Gauss-Doolittle reduction
- Cholesky reduction
Eigenvalue Problems

Many problems require that two matrices $[A]$ & $[B]$ need to be reduced

$$[A] \{ \dot{x} \} + [B] \{ x \} = \{ Q(t) \} \quad \rightarrow \quad \left[ [B] - \lambda [A] \right] \{ x \} = 0$$

Applications for solution of eigenproblems are plentiful. Common methods are

- Jacobi
- Givens
- Householder
- Subspace Iteration
- Lanczos
**Singular Valued Decomposition**

Any matrix can be decomposed using SVD

\[
[A] = [U][S][V]^T
\]

[U] - matrix containing left hand eigenvectors

[S] - diagonal matrix of singular values

[V] - matrix containing right hand eigenvectors
Singular Valued Decomposition

SVD allows this equation to be written as

$$[A] = [\{u_1\} \{u_2\} \{u_3\} \cdots] \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots \end{bmatrix} [\{v_1\}^T \{v_2\}^T \{v_3\}^T \cdots]$$

which implies that the matrix $[A]$ can be written in terms of linearly independent pieces which form the matrix $[A]$

$$[A] = \{u_1\} s_1 \{v_1\}^T + \{u_2\} s_2 \{v_2\}^T + \{u_3\} s_3 \{v_3\}^T + \cdots$$
Singular Valued Decomposition

Assume a vector and singular value to be

\[
u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad s_1 = 1
\]

Then the matrix \([A_1]\) can be formed to be

\[
[A_1] = \{u_1\} s_1 \{u_1\}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}
\]

The size of matrix \([A_1]\) is (3x3) but its rank is 1.
There is only one linearly independent piece of information in the matrix.
Singular Valued Decomposition

Consider another vector and singular value to be

\[ u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad s_2 = 1 \]

Then the matrix \([A_2]\) can be formed to be

\[
\begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
-1 & -1 & 1
\end{bmatrix}
\]

The size and rank are the same as previous case

Clearly the rows and columns are linearly related
Singular Valued Decomposition

Now consider a general matrix $[A_3]$ to be

$$[A_3] = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 5 & 5 \\ 2 & 5 & 10 \end{bmatrix} = [A_1] + [A_2]$$

The characteristics of this matrix are not obvious at first glance.

Singular valued decomposition can be used to determine the characteristics of this matrix.
Singular Valued Decomposition

The SVD of matrix $[A_3]$ is

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$[A] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

These are the independent quantities that make up the matrix which has a rank of 2.
The basic solid mechanics formulations as well as the individual elements used to generate a finite element model are described by matrices.
Linear Algebra Applications

Finite element model development uses individual elements that are assembled into system matrices.
**Linear Algebra Applications**

**Structural system equations - coupled**

\[
[M]{\ddot{x}} + [C]{\dot{x}} + [K]{x} = \{F(t)\}
\]

**Eigensolution - eigenvalues & eigenvectors**

\[
[[K] - \lambda[M]]{x} = 0
\]

**Modal space representation of equations - uncoupled**

\[
\begin{bmatrix}
\bar{M} \\
\bar{C} \\
\bar{K}
\end{bmatrix}
\begin{bmatrix}
\tilde{p} \\
\tilde{p} \\
\tilde{p}
\end{bmatrix}
+ \begin{bmatrix}
\bar{M} \\
\bar{C} \\
\bar{K}
\end{bmatrix}
\begin{bmatrix}
\tilde{p} \\
\tilde{p} \\
\tilde{p}
\end{bmatrix}
= \begin{bmatrix}
{U}^T \\
{U}^T \\
{U}^T
\end{bmatrix}
\begin{bmatrix}
\tilde{F}
\end{bmatrix}
\]
Linear Algebra Applications

Multiple Input Multiple Output Data Reduction

$$[G_{yx}] = [H][G_{xx}] \quad \Rightarrow \quad [H] = [G_{yx}][G_{xx}]^{-1}$$

Matrix inversion can only be performed if the matrix $[G_{xx}]$ has linearly independent inputs.
**Linear Algebra Applications**

**Principal Component Analysis using SVD**

\[
\begin{bmatrix}
{G_{xx}} \\
\end{bmatrix} = \begin{bmatrix}
{u_1} & {u_2} & {0} & \cdots \\
\end{bmatrix} \begin{bmatrix}
S_1 & & & \\
S_2 & 0 & \cdots & \\
\end{bmatrix} \begin{bmatrix}
\{v_1\}^T \\
\{v_2\}^T \\
\{0\}^T \\
\vdots \\
\end{bmatrix}
\]

**SVD of the input excitation matrix identifies the rank of the matrix - that is an indication of how many linearly independent inputs exist**
**Linear Algebra Applications**

**SVD of Multiple Reference FRF Data**

\[
[H] = [\{u_1\} \ {u_2} \ {u_3} \ \ldots] \begin{bmatrix}
S_1 & 0 & 0 \\
0 & S_2 & 0 \\
0 & 0 & S_3 \\
\end{bmatrix}
\begin{bmatrix}
\{v_1\}^T \\
\{v_2\}^T \\
\{v_3\}^T \\
\end{bmatrix}
\]

**SVD of the [H] matrix gives an indication of how many modes exist in the data**

**FREQUENCY RESPONSE FUNCTIONS**
Least Squares or Generalized Inverse for Modal Parameter Estimation Techniques

\[
H(s) = \sum_{k=i}^{j} \frac{[A_k]}{(s-S_k)} + \frac{[A^*_k]}{(s-S^*_k)}
\]

Least squares error minimization of measured data to an analytical function
Linear Algebra Applications

Extended analysis and evaluation of systems

\[
\begin{align*}
\mathbf{K}[\mathbf{U}] &= \mathbf{M}_1[\mathbf{U}]\omega^2 \\
\mathbf{U}^T[\mathbf{K}_1][\mathbf{U}] &= \mathbf{U}^T[\mathbf{M}_1][\mathbf{U}]\omega^2 \\
\mathbf{K}_1 &= \mathbf{K}_S + [\mathbf{V}]^T\left[\omega^2 + \mathbf{K}_S\right][\mathbf{V}] \\
&\quad - \left[\mathbf{K}_S[\mathbf{U}]\mathbf{U}^T[\mathbf{M}_1]\right] - \left[\mathbf{K}_S[\mathbf{U}]\mathbf{U}^T[\mathbf{M}_1]\right]^T \\
\mathbf{K}_1 &= \mathbf{K}_S + [\mathbf{V}]^T\left[\omega^2 + \mathbf{K}_S\right][\mathbf{V}] - \left[\mathbf{K}_S[\mathbf{U}]\mathbf{V}\right] - \left[\mathbf{K}_S[\mathbf{U}]\mathbf{V}\right]^T
\end{align*}
\]

generally require matrix manipulation of some type
Linear Algebra Applications

Many other applications exist

Correlation  Model Updating
Advanced Data Manipulation
Operating Data  Rotating Equipment
Nonlinearities
Modal Parameter Estimation

and the list goes on and on