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Natural Constants Determined to High Precision from Boltzmann's Constant and Avogadro's Number—A Challenge to Experiments and Astrophysical Observations to Match the Precision of the Results

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Abstract

In this investigation, we explore previously unknown relations between natural constants by taking the following steps: (1) We discard Dirac's constant $\hbar$ from the universal man-made constants of physics, which we redefine in terms of Planck's constant h. (2) Working in the SI system of units, we determine Newton's gravitational constant G from Boltzmann's constant k_B and the elementary charge e, recognizing the entropy of matter as their common underlying characteristic. (3) By comparing the mass of 1 mole of electrons to the h-defined Planck mass M_P, we deduce nature's own molar constant ($\simeq$0.1 mol) that contains a 'reduced Avogadro number' $\aleph_A = N_A/f_A$ of particles, where N_A is Avogadro's number and $f_A \simeq 10$ is the associated Avogadro factor. (4) From the new effective gravitational constant $G_\star \equiv 4\pi\varepsilon_0 G$, where ε_0 is the vacuum permittivity, we obtain MOND's universal constant $\mathcal{A}_0$ and its critical acceleration a_0, recognizing the Newtonian source of gravity as the common underlying characteristic and repudiating the need for a principle of equivalence of masses. (5) We derive the gravitational coupling constant α_g solely from $\aleph_A$. (6) We adopt the measured value of the h-defined fine-structure constant (FSC) α and the value of α_g (or, equivalently, nature's $\aleph_A$), and we determine the relative ratio $\beta_g = \alpha_g/\alpha$ precise to 10 significant digits. (7) We derive the relative strong ratio $\beta_s = \alpha_s/\alpha$ directly from the Avogadro factor f_A. (8) We determine the coupling constants of weak and strong interactions (α_w and α_s, respectively) in terms of the FSC α. (9) The relation $\alpha_w = \sqrt{\alpha}$ leads to a determination of the mass of the W boson m_W from the measured values of α and the reduced Fermi constant G_F^0. (10) Using the Planck mass as a principal constant ($M_P = \aleph_A m_e$, where m_e is the electron mass), we obtain new classical definitions of h, α, and the Compton radius r_c; and we reformulate in a transparent, geometrically clear way several important QED equations, as well as the extended Planck system of units itself. We discuss the implications of these results, and we pave a way forward in exploring the unification of the fundamental forces of nature.

Keywords: cosmology; coupling constants; fine-structure constant; gravitation; metrology; MOND theory; Planck system of units

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1. Introduction

In this work, we present a set of relations between fundamental physical constants across all scales and disciplines. We also correct and interrelate the dimensionless coupling constants of the four fundamental forces, and we reformulate the Planck system of units in a transparent way that provides straightforward interpretations of the quantities involved and their dependencies. These results could not have been obtained in the past; they emerge only now after correcting certain century-old conceptual errors that have crept into the foundations of physics, distorting our man-made (subjective) definitions and our interpretations of the empirical force laws confirmed by experiments.

In the following two subsections, we describe these ingrained conceptual pitfalls and highlight the significant benefits arising directly from corrective actions (Sections 1.1 and 1.2, respectively). Section 1 wraps up with an outline of the remaining sections of the paper (Section 1.3).

1.1. Conceptual Pitfalls

1.1.1. Dirac's Constant $\hbar$, Our Inconspicuous Nemesis

In a scientific conference circa 1930, Paul Dirac proclaimed without explanation that the true universal constant was not Max Planck's h [1,2], but instead the 'reduced constant' $\hbar = h/(2\pi)$ (see, e.g., [3,4]). No-one asked for an elaboration, the physicists in attendance must have thought that Dirac was conveniently simplifying the known equations of quantum mechanics by absorbing the 2π into Planck's h, something that Erwin Schrödinger [5] also did at about the same time, calling the new constant K, but without any further assertion or declaration as to its physical significance.

But Dirac had a higher goal in mind[1] than a mere simplification—and people went along with his idea for no counter-arguments were brought forth, until now. Sadly, by elevating $\hbar$ to universal status, the 2π imprint of two-dimensional (2-D) geometry attached to Planck's h disappeared from plain view forever. This miscue has since permeated the backbone of the physical sciences, causing gross misinterpretations of many fundamental constants featuring $\hbar$, a composite constant that always carries along an invisible tag of 2-D geometry [6].

Equations containing three-dimensional (3-D) geometric dependencies (4π terms), such as the fine-structure constant (FSC), lose their meaning due to odd combinations of disparate geometries; and pure 3-D equations, such as the Planck units, are erroneously imprinted with the unit of radian; although the presence of 2-D geometry cannot be detected since radians have been dropped from the SI unit of $\hbar$ by international agreement.

Yet, certain facets of the problem were recently reported by Bunker et al. [7] who asked for radians to be reinstated in $\hbar$, and by Leblanc et al. [8] who showed that the Compton radius of the electron $r_c = \hbar/(m_e c)$ (where m_e is the mass and c is the speed of light) in not a purely physical constant, since it oddly includes a geometric component. In contrast, there is no geometric imprint in the de Broglie wavelength $\lambda = h/(m_e v)$ [9] of an electron moving at speed v; all physical quantities are understood as being intrinsically three-dimensional, and no geometric term is needed to be included.

These important works did not succeed in exposing and clarifying the composite nature of $\hbar$ in physics. The general perception is that using $\hbar$ instead of h is beneficial, despite the fact that the additional justifications sought to strengthen this perception after the fact (e.g., [10]) can be patently rebutted and likely refuted.

1.1.2. The Two Geometric Means of Two Natural Constants

It is well-known, though certainly underrated in the atomic world (the speed of light is not a unit in the atomic system of units [11]), that the vacuum establishes rules in its domain; these are the lower limits known as vacuum permittivity ε_0 and vacuum permeability μ_0. In an impartial ("fair") vacuum that contemplates its properties equitably, the geometric-mean (G-M) relations of these properties should be present as well [6].

Thus, the unbiased combinations of ε_0 and μ_0 produce two ubiquitous thresholds known as the speed of light c (an upper limit by construction) and the impedance of free space Z_0 (not a limit, just a threshold). From their G-M definitions [12–14], viz.

$$c = \sqrt{\varepsilon_0^{-1}\mu_0^{-1}} \quad \text{and} \quad Z_0 = \sqrt{\varepsilon_0^{-1}\mu_0}\,, \tag{1}$$

it seems that the vacuum establishes these four properties, and then it sits back, a mere observer of interactions between fields and particles that fill some of its space—which however must conform to the imposed rules.

On the other hand, certain details concerning the four vacuum constants are not at all clear. Specifically, (a) the speed of light is thought to be fundamental in nature although its G-M derivation indicates that c is simply a derivative; (b) whereas Z_0 has never achieved the same recognition as c; and (c) the fact that, besides c, the other three vacuum constants never appear in 3-D problems without an attached 3-D geometric factor of 4π has gone unnoticed for more than a century[2].

1.1.3. The Dimensional Constants of the Two Long-Range Force Fields

The appearance of the composite constant $1/(4\pi\varepsilon_0)$ in Coulomb's law has also gone unnoticed for more than a century, but it is a cause for concern. The problem is that the vacuum does not seem to impose the same constraint to Newton's gravitational law in which the normalizing constant G is not regulated by the vacuum and is not tagged by a 3-D geometric factor.

The obvious difference between these two fundamental laws should have rung a bell long ago. Instead, we teach our young to marvel at the amazing similarity between these two conservative long-range forces without paying attention to the attached constants. More than that, we have called "Coulomb's constant" K the $1/(4\pi\varepsilon_0)$ factor in Coulomb's law [13,14], burying the influence of the vacuum and achieving our hearts' desire, a "truly wonderful similarity" between the two conservative forces ($F = GM_1M_2/r^2$ and $F = KQ_1Q_2/r^2$). Talk about the wrong substitution!

Textbooks are silent on the principles of substitutions as applied to equations, inequalities, and expressions. The general consensus dictates that one can substitute any name for anything—after all, it is merely a renaming act. Here and in Section 1.1.1, we have demonstrated that substitutions lay out veils that conceal composite factors, and their compositions and internal properties may then be quickly forgotten.

1.1.4. The Centuries-Old Conundrum with Systems of Units

Physical insight has been repeatedly sacrificed in the name of convenience. Besides the SI system of units, all other systems have been created with the intention of setting various constants equal to 1 [12]. This common practice is predominantly useful in theoretical studies, which however are unable to produce actual numerical values, in which case researchers and authors back-pedal by restoring all units to their true forms. The SI system of units of measurement has become dominant precisely because it does not hide or suppress natural constants.

As will be seen below, setting the composite constant $\hbar = 1$ has caused the largest losses in insightful physical interpretations, although historically the debates about setting $\varepsilon_0 = 1$ or $4\pi\varepsilon_0 = 1$ have also resulted in quite a few confusing arguments over many years [12,15].

1.1.5. The Man-Made Unit of 1 Mole and Avogadro's Number N_A

The SI unit of 1 mole has always played a central role in chemical measurements as well as in theoretical calculations [16], so much so that people are no longer concerned with its arbitrary character and the closely associated number of constituent particles expressed by Avogadro's number N_A. In the SI system, N_A is now taken to be 'exact' with 9 significant digits (SDs) [13,14].

These seemingly acceptable and internationally accepted definitions are not only arbitrary (like many other SI constants), but they also turn out to be the source of many problems in physical theory (unlike many other SI constants and like the definition of $\hbar$). The reason for this unsuspected conundrum is that nature does not recognize or support the mole as one of her constants (unlike many other SI constants, such as speed c, charge e, and mass m_e).

This damaging problem can only be resolved by finding nature's own 'molar unit' and the corresponding 'reduced Avogadro number' of particles (see Section 1.2.2 below).

1.1.6. Dimensionless Coupling Constants, a Century-Old Stumbling Block

Physical theory has defined four (man-made) coupling constants, one for each fundamental force of nature, and metrology has set out to measure experimentally three of them that are not negligibly small: the electromagnetic (EM) FSC α, the weak coupling constant α_w, and the strong coupling constant α_s. On the other hand, the gravitational coupling constant α_g is calculated from other measured constants.

The current definitions of α and α_g are incorrect because Dirac's composite $\hbar$ is used instead of Planck's purely physical 3-D constant h. Below, we define these two constants in terms of Planck's h, and the resulting adjusted values become clear and easy to interpret physically, thereby overturning numerous defeatist suppositions born of inadequate data in the published literature.

1.2. Significant Benefits

1.2.1. Understanding the Vacuum Constants $4\pi\varepsilon_0$, $\mu_0/(4\pi)$, and $Z_0/(4\pi)$

Vacuum permittivity ε_0 and permeability μ_0 appear to be the two fundamental (minimum) constants introduced by the vacuum, although they are always imprinted by a factor of 4π[3]. The G-M derivatives c and $Z_0/(4\pi)$ are also introduced by the vacuum (Section 1.1.2). The factors of 4π cancel out in the derivation of the speed of light, but not in the derivation of the impedance of free space, which then must always be considered in the form $Z_0/(4\pi)$. By construction, the speed of light describes a kinematic upper limit independent of the number of dimensions and valid in all possible directions.

The importance of the composite constant $Z_0/(4\pi)$ cannot be understated. Not only does it always carry a factor of 4π, but it also has a robust physical significance in 3-D space (just like the quantum of angular momentum $\hbar$ does in 2-D space). A simple calculation [6] using the definitions of vacuum-related quantities provided by Refs. [12,15,20] shows that $\sqrt{\mu_0/\varepsilon_0}/(4\pi) = 1/(4\pi\varepsilon_0 c)$, or, equivalently, that

$$\frac{Z_0}{4\pi} = \mathcal{R}_P\,, \tag{2}$$

where $\mathcal{R}_P \equiv K/c$ is the unit of electric resistance in the Planck system [20] and $K \equiv 1/(4\pi\varepsilon_0)$ is Coulomb's constant.

Thus, we can describe self-consistently a set of six vacuum constants that are infused to matter, energy, particles, and fields when they materialize in the vacuum:

$$\text{Universal vacuum constants} := \left\{ \varepsilon_0\,,\ \mu_0\,,\ 4\pi\varepsilon_0\,,\ \frac{\mu_0}{4\pi}\,,\ \frac{Z_0}{4\pi}\,,\ c \right\}. \tag{3}$$

We have included ε_0 and μ_0 in this set because these terms appear to regulate alone the sources of EM fields, where formal geometric constraints require the 2π or 4π terms to be produced by line or volume integrations, respectively (Ampére's law [21], Gauss's law [22], and Maxwell's equations [23,24]).

1.2.2. Discovering Nature's Own Molar Unit and Its Number of Particles $\aleph_A$

The SI unit of 1 mole encapsulates the number and the cumulative mass of a group of like particles for which individual particle masses are measured experimentally. Unfortunately, this unit is arbitrary, and we can be certain that nature does not subscribe to it. This fact is responsible for our inability to connect certain fundamental constants, although some progress has been made recently (Ref. [6] and this work). Probably the only way of resolving such issues is to relate the unit of 1 mole and Avogadro's number N_A to physical constants that do not depend on our subjective choices, such as the (h-defined) units of the Planck system and the (h-defined) dimensionless constants that have not been included in any system of units [2,20,25–28].

In the following sections, we carry out such comparisons between units that have not been heretofore possible because of the erroneous use of Dirac's $\hbar$ in 3-D physical settings. This exercise yields immediate benefits:

(a) The reduced Avogadro number $\aleph_A$ corresponding to nature's molar unit is determined in two different ways.
(b) The gravitational coupling constant α_g is determined solely in terms of $\aleph_A$, a long-sought hypothesized connection.
(c) The Planck mass M_P is directly proportional to the electron mass m_e, a feat previously thought to be unfeasible [25].
(d) Several universal constants are found to be derivatives, thereby resolving the conundrum concerning which constants are truly fundamental in nature [25–28].
(e) The Planck system of units [2,20] is reformulated in simple and clear terms with distinct composite constants (Section 1.2.4) describing static and moving charges (electric currents).

1.2.3. Relating the Coupling Constants of the Four Fundamental Forces

The ratio of Avogadro's number N_A to the reduced value $\aleph_A$ is a new universal constant that we call the Avogadro factor f_A. Its value is $f_A \simeq 10$, so nature's own molar unit is about 0.1 mol. This determination is obtained from α_g or from M_P and, independently, from the relative strong coupling ratio $\beta_s = \alpha_s/\alpha$; and yields relations between the coupling constants of the four fundamental forces. Below we express these constants in terms of the (h-defined) FSC which is currently known to a precision of 11 SDs from its inverse ($\alpha^{-1} \simeq 861$; Ref. [6]) measured to 12 SDs ([29]; PDG Refs. [30,31]; CODATA Refs. [13,14]).

1.2.4. Defining Two Convenient Vacuum-Tagged Effective Gravitational Constants $G_\star$ and G_B

In a surprising series of derivations:

(1) The SI numerical value of Newton's gravitational constant G is found to be a derivative of the constants e (elementary charge), $4\pi\varepsilon_0$ (vacuum permittivity), and k_B (Boltz-

mann's constant). This shows that G carries information about the entropy of the gravitational field.

(2) We define two effective gravitational constants imprinted by vacuum EM constants, viz.

$$G_\star \equiv (4\pi\varepsilon_0)G\,, \tag{4}$$

and

$$G_B \equiv \left(\frac{\mu_0}{4\pi}\right)G\,. \tag{5}$$

They indicate that the vacuum coupled to G may act on gravitational fields as well as on EM and QED fields, a property that becomes evident in the reformulated Planck system of units.

(3) Constant G_B signifies the presence of moving charges and electric currents.

(4) Constant $G_\star$ scales the source of the gravitational field $G_\star M$ produced by an inertial mass M and dispenses with the need for an equivalence principle of masses [32].

(5) The strength (numerical value) of $G_\star$ naturally scales the source of gravity in MOND as well. In particular,

$$\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(G_\star)\,, \tag{6}$$

where $\mathcal{A}_0$ is MOND's universal constant [33,34] and the numerical function $\mathcal{N}(\cdot)$ indicates that units are set aside. Furthermore, $\mathcal{N}(a_0) = \mathcal{N}(4\pi\varepsilon_0)$, where a_0 is MOND's critical acceleration. These relations indicate that the constants of MOND do not have a cosmological origin. The MOND constants are discussed in detail in Appendix A.1.

(6) Important QED constants (such as Planck's h, the Compton radius r_c, and the FSC α) are found to have a classical origin (perhaps even a 'gravitational' origin). They are all expressed in terms of the composite constant $G(\aleph_A)^2$, but they assume their simplest forms when written in terms of the classical Planck mass $M_P = \aleph_A m_e$.

1.2.5. Understanding the Stoney Mass M_S and Length L_S

The Stoney units of mass and length [28] are obtained easily from the two G-Ms of the constants e^2 and $G_\star$. This reasoning that uses G-Ms is relatively recent, but it has led to new, previously undetected physical relations (Refs. [6,35,36] and this work).

The Stoney mass M_S is discussed in detail in Appendix A.2. It shows that the source term of the gravitational field $G_\star M_S \propto k_B$, whereas the electron's gravitational source term $G_\star m_e \propto k_B/\aleph_A$.

The Stoney length L_S has not been previously appreciated despite exhibiting two important properties: (a) $L_S \equiv R_e$, where R_e is the charge radius of the electron [37,38], a length scale that also appears in Reissner-Nordström black-hole physics [39–46]; and (b) $\mathcal{N}(L_S) = \mathcal{N}(k_B) \times 10^{-13}$, indicating that entropy information encoded into $G_\star$ is passed on to a linear setting with just one degree of freedom.

Constants $G_\star$ and G_B ($\propto G_\star$) introduce entropy considerations into the EM section of the Planck system of units, whereas Newton's G (also $\propto G_\star$) is the carrier of such information in the mechanical section of the reformulated system. Details are given in Section 5 below.

1.2.6. Discovering the Weak Interaction and a New Natural Charge $Q_\star$

The ratio of charge e to the Planck charge Q_P produces the weak coupling constant $\alpha_w \simeq 0.034$. This result may be surprising, although the individual constants are well-known. In our formulation that uses the h-defined FSC, such a relation was expected because experimental measurements show clearly that $\alpha_w = \sqrt{\alpha}$.

On the other hand, the G-M of the two charges yields a brand-new charge scale, viz.

$$Q_\star = \sqrt{e\,Q_{\rm P}}\,. \tag{7}$$

The three charges form the geometric sequence $\{e,\, Q_\star,\, Q_{\rm P}\}$ with common ratio

$$\frac{Q_{\rm P}}{Q_\star} = \frac{Q_\star}{e} = {\alpha_{\rm w}}^{-1/2} \simeq 5.4169\,. \tag{8}$$

Besides the appearance of $\alpha_{\rm w}$ as an electroweak constant, the significance of the $Q_\star$ scale is currently not fully understood.

1.3. Outline

The remainder of the paper is organized as follows:

- In Section 2, we present in tabular form the calculations that demonstrate numerous relations and dependencies between various universal constants.
- In Section 3, we present the calculations that determine various quantum mechanical constants from other classical constants.
- In Section 4, we analyze the geometric imprints in various proposed QED equations in which Dirac's $\hbar$ has been used routinely.
- In Section 5, we present the reformulated Planck system in a simple and concise form based on our choice of 7 fundamental (field+vacuum+molar) constants, i.e., $\{e,\ m_{\rm e},\ k_{\rm B};\ \ \varepsilon_0,\ \mu_0;\ \ N_{\rm A},\ f_{\rm A}\}$.
- In Section 6, we discuss briefly our results and summarize our conclusions.
- In Appendix A.1, we summarize additional information concerning MOND's universal constant $\mathcal{A}_0$ [33,34], its critical acceleration a_0, and the source term $\mathcal{A}_0 M$ of the gravitational field due to a mass M.
- In Appendix A.2, we discuss the G-M of e^2 and ${G_\star}^{-1}$ known as the Stoney mass [28], and the source term $G_\star m_{\rm e}$ of the gravitational field of the electron.
- In Appendix A.3, we provide summary tables in which we collect and categorize all the relations between constants and units discussed in this paper.

2. Relations Involving Universal Constants

We explore relations between well-known universal constants. We derive only one new constant, the Avogadro factor $f_{\rm A}$ that scales Avogadro's number $N_{\rm A}$ to the reduced natural value $\aleph_{\rm A} = N_{\rm A}/f_{\rm A}$. The value of $f_{\rm A} \simeq 10$ is determined in two different ways (from the Planck mass and the strong coupling constant) which dispels notions of a mere coincidence.

The results are presented in tabular form, and they are summarized in the 11 tables that follow. Each table lists the measured input parameters at the top and the derived constants at the bottom. The results are separated from the input parameters by a horizontal line in each table. Notes below each table report on the details of the measurements and the calculations.

Most of the results (including Newton's G, nature's $\aleph_{\rm A}$, and Planck's $M_{\rm P}$) are given to a precision of at least 10 SDs. Only the mass of the W boson is reported to 7 SDs because the input reduced Fermi constant $G_{\rm F}^0$ is currently measured to 8 SDs [14,31].

2.1. Table 1

Table 1 shows the initial discovery that $\mathcal{N}(k_{\rm B,MeV}) = \mathcal{N}(\sqrt{G_\star})$. The entropy constant $k_{\rm B,MeV}$ is Boltzmann's constant measured in MeV. The elementary charge e does not appear because of the chosen units (but see Table 2 for an alternative calculation using different units).

Table 1. $\mathcal{N}(k_{\rm B,MeV}) = \mathcal{N}(\sqrt{4\pi\varepsilon_0 G})$, a numerical identity rooted to entropy that shows nature imprinting the same property in different settings irrespective of units and their (sub)multiples *.

Constant	Symbol	Value	SDs	SI Unit	Source
Vacuum Permittivity	ε_0	$8.854\,187\,8188(14) \times 10^{-12}$	11	$\rm F\,m^{-1}$	CODATA
Gravitational Constant	G	$6.674\,30(15) \times 10^{-11}$	6	$\rm m^3\,kg^{-1}\,s^{-2}$	CODATA
Boltzmann Constant	$k_{\rm B,MeV}$	$8.617\,333\,262 \times 10^{-11}$	Exact	$\rm MeV\,K^{-1}$	CODATA
G-M of $4\pi\varepsilon_0$ and G	$\sqrt{4\pi\varepsilon_0 G}$	$8.617\,5(1) \times 10^{-11}$ **	5	$\rm C\,kg^{-1}$	Calculated

* The SI system of units and their derivatives are used throughout this analysis; all other systems of units suffer from inconsistencies introduced for the sake of simplicity (e.g., $G = 1$, $c = 1$, $4\pi\varepsilon_0 = 1$, and so on). ** We attribute the 19.35-ppm difference to the poorer measurements of G, and we use Boltzmann's constant to obtain a more precise value of G in Table 2 below.

2.2. Table 2

Table 2 shows the calculation of Newton's G from $G_\star$ which was obtained from the more precise Boltzmann's constant and the elementary charge using different sets of units. The numerical calculations of the results are carried out by the formulae shown in the 'Source' column. The full equations are obtained by equating each listed 'Symbol' to its 'Source,' e.g., $\mathcal{N}(G_\star) = \mathcal{N}(k_{\rm B,MJ}/e)^2$ and $G = G_\star/(4\pi\varepsilon_0)$. The value of G is determined to a precision of 10 SDs and is compared to experimental results in the notes to the table.

Table 2. CODATA universal constants are used to determine to a high precision of 10 SDs the gravitational constants $G_\star \equiv 4\pi\varepsilon_0 G$ (effective) and G (Newtonian) shown at the bottom.

Constant	Symbol	Value	SDs	SI Unit	Source
Vacuum Permittivity	ε_0	$8.854\,187\,8188(14) \times 10^{-12}$	11	$\rm F\,m^{-1}$	CODATA
Boltzmann Constant	$k_{\rm B,MeV}$	$8.617\,333\,262 \times 10^{-11}$	Exact	$\rm MeV\,K^{-1}$	CODATA
	$(k_{\rm B,MeV})^2$	$7.425\,843\,255 \times 10^{-21}$	Exact	$\rm MeV^2\,K^{-2}$	Calculated
Elementary Charge	e	$1.602\,176\,634 \times 10^{-19}$	Exact	C	CODATA
Boltzmann Constant	$k_{\rm B,MJ}$	$1.380\,649 \times 10^{-29}$	Exact	$\rm MJ\,K^{-1}$	CODATA
	$(k_{\rm B,MJ}/e)^2$	$7.425\,843\,255 \times 10^{-21}$	Exact	$\rm MJ^2\,K^{-2}\,C^{-2}$	Calculated
Effective Grav. Constant	$G_\star$	$7.425\,843\,255 \times 10^{-21}$	Exact *	$\rm C^2\,kg^{-2}$	$\mathcal{N}(k_{\rm B,MeV})^2$ or $\mathcal{N}(k_{\rm B,MJ}/e)^2$
Gravitational Constant	G	$6.674\,015\,081(1) \times 10^{-11}$ **	10	$\rm m^3\,kg^{-1}\,s^{-2}$	$G_\star/(4\pi\varepsilon_0)$

* We adopt $G_\star$ as exact since Boltzmann's constant and the elementary charge are exact in the SI system. ** The 2018–2022 world average $6.674\,30(15) \times 10^{-11}$ $\rm m^3\,kg^{-1}\,s^{-2}$ [13,14] matches the first 4 error-free SDs. The average of CODATA recommended values 1998–2022 matches 5 SDs. One of four recent high-precision experiments [47,48] also matched 5 SDs.

2.3. Table 3

Table 3 shows the calculations of the Planck units of mass, length, and charge using their standard definitions, but with Planck's constant h restored in place of $\hbar$. These numerical values differ from those quoted in the literature by factors of $\sqrt{2\pi} \simeq 2.5$.

Table 3. CODATA universal constants and Newton's G from Table 2 are used to determine the original Planck units of mass M_P, length L_P, and charge Q_P shown at the bottom *.

Constant	Symbol	Value	SDs	SI Unit	Source
Vacuum Permittivity	ε_0	$8.854\,187\,8188 \times 10^{-12}$	11	F m^{-1}	CODATA
Light Speed	c	$2.997\,924\,58 \times 10^{8}$	Exact	m s^{-1}	CODATA
Planck Constant	h	$6.626\,070\,15 \times 10^{-34}$	Exact	J Hz^{-1}	CODATA
Gravitational Constant	G	$6.674\,015\,081 \times 10^{-11}$	10	m^3 kg^{-1} s^{-2}	Table 2
Planck Mass	M_P	$5.455\,628\,310 \times 10^{-8}$	10	kg	$\sqrt{hc/G}$
Planck Length	L_P	$4.051\,264\,068 \times 10^{-35}$	10	m	$\sqrt{hG/c^3}$
Planck Charge **	Q_P	$4.701\,296\,730 \times 10^{-18}$	10	C	$\sqrt{4\pi\varepsilon_0 hc}$

* We emphasize that Planck's 3-D universal constant h is implemented in all man-made definitions instead of the erroneous 2-D constant $\hbar$, whereas $\hbar$ is replaced by $h/(2\pi)$ in all natural equations in which the presence of the 2π is justified. ** The Planck charge Q_P is intimately connected to the elementary charge e in various ways: (a) the G-M of e^2 and $(1/Q_P)^2$ is the weak coupling constant α_w, leading to the relation $\alpha_w = e/Q_P$; (b) the other G-M of e^2 and $Q_P{}^2$ with dimensions of [charge]^2 leads to the relation $Q_\star^2 = eQ_P$, so a new charge $Q_\star$ is the G-M of e and Q_P and $Q_\star = 8.678\,886\,893 \times 10^{-19}$ C; (c) the charges $\{e, Q_\star, Q_P\}$ form a geometric progression with common ratio $1/\sqrt{\alpha_w} \simeq 5.4169$. Furthermore, the following relations hold for these charges: $Q_\star^2 = M_P\sqrt{e^2 G_\star}$ and $Q_P = M_P\sqrt{G_\star}$, which also imply the numerical identities $\mathcal{N}(Q_\star)^2 = \mathcal{N}(M_P)\mathcal{N}(k_{B,MJ})$ and $\mathcal{N}(Q_P) = \mathcal{N}(M_P)\mathcal{N}(k_{B,MeV})$, respectively.

2.4. *Table 4*

Table 4 shows the calculations of the Avogadro factor f_A and the reduced Avogadro number $\aleph_A$ for which $\aleph_A$ electrons have a cumulative mass equal to the Planck mass determined in terms of Planck's h (that is, $\aleph_A m_e = M_P$, where $M_P = \sqrt{hc/G}$). The derived natural values of f_A and $\aleph_A$ are not arbitrary; they also determine the relative strong coupling constant $\beta_s = \alpha_s/\alpha$ and the gravitational coupling constant $\alpha_g = Gm_e^2/(hc)$, respectively.

Table 4. CODATA universal constants and Planck mass M_P from Table 3 are used to determine the Avogadro factor f_A and the reduced Avogadro number $\aleph_A$ shown at the bottom *. Then, an antithesis is formulated in the last row of the table: the fundamental constants $\aleph_A$ and m_e are used to define the Planck mass in a novel way.

Constant	Symbol	Value	SDs	SI Unit	Source
Avogadro Number	N_A	$6.022\,140\,76 \times 10^{23}$	Exact	—	CODATA
Electron Mass	m_e	$9.109\,383\,7139 \times 10^{-31}$	11	kg	CODATA
Mass of 1 mole of Electrons	$N_A m_e$	$5.485\,799\,0962 \times 10^{-7}$	11	kg	Calculated
Planck Mass	M_P	$5.455\,628\,310 \times 10^{-8}$	10	kg	Table 3
Avogadro Factor	f_A	$10.0553\,0213$	Exact	—	$N_A m_e/M_P$
Inverse Avogadro Factor	f_A^{-1}	$0.099\,450\,020\,21$	Exact	—	Calculated
Reduced Avogadro Number **	$\aleph_A$	$5.989\,020\,203 \times 10^{22}$	Exact	—	N_A/f_A
Planck Mass (new definition)	M_P	$5.455\,628\,310 \times 10^{-8}$	10	kg	$\aleph_A m_e$

* We adopt the derived constants as exact to align with Avogadro's number N_A which is exact in the SI system. ** The reduced Avogadro number determines the gravitational coupling constant ($\alpha_g = 1/(\aleph_A)^2$) and the Planck mass ($M_P = \aleph_A m_e$) which, in turn, determines many other universal constants (including $h = GM_P^2/c$) and helps reformulate the Planck system of units in the simple form presented in Section 5 below.

2.5. *Table 5*

Table 5 establishes the identities $\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(G_\star)$ and $\mathcal{N}(a_0) = \mathcal{N}(4\pi\varepsilon_0)$ for the MOND constants $\mathcal{A}_0$ and a_0, respectively. In retrospect, the magnitudes of $\mathcal{A}_0$ and $G_\star$ could not have been different because they both appear in the source of the gravitational field ($G_\star M$ in Newtonian gravity and $\mathcal{A}_0 M$ in MOND for a mass M). Thus, the source of gravity has the same strength in the two regimes, although the MOND force is modified on the whole by a square root [49], an action that changes the units of $\mathcal{A}_0$[4].

Table 5. The effective gravitational constant $G_\star$ from Table 2 is used to determine the MOND constants shown at the bottom *.

Constant	Symbol	Value	SDs	SI Unit	Source
Effective Grav. Constant	$G_\star$	$7.425\,843\,255 \times 10^{-21}$	Exact	$C^2\ kg^{-2}$	Table 2
Vacuum Permittivity	ε_0	$8.854\,187\,8188(14) \times 10^{-12}$	11	$F\ m^{-1}$	CODATA
MOND Universal Constant	$\mathcal{A}_0$	$7.425\,843\,255 \times 10^{-21}$	Exact	$m^4\ kg^{-1}\ s^{-4}$	$\mathcal{N}(G_\star)$
MOND Critical Acceleration	a_0	$1.112\,650\,0562(18) \times 10^{-10}$	11	$m\ s^{-2}$	$\mathcal{N}(4\pi\varepsilon_0)$

* The identities $\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(G_\star)$ and $\mathcal{N}(a_0) = \mathcal{N}(4\pi\varepsilon_0)$ indicate that the MOND constants are determined by gravity and the vacuum; thus, they are not connected to cosmological constants as previously thought. Some of the conclusions obtained from these numerical equalities are discussed in Appendix A.1.

2.6. *Table 6*

Table 6 shows two determinations of the gravitational coupling constant α_g. The conventional definition uses Planck's h (instead of $\hbar$), but it no longer seems to be fundamental or have a QED origin since α_g can now be tied directly to the reduced Avogadro number $\aleph_A$.

Table 6. Gravitational constant G from Table 2 and reduced Avogadro number $\aleph_A$ from Table 4 are used to determine the gravitational coupling constant α_g in two different ways * shown at the bottom.

Constant	Symbol	Value	SDs	SI Unit	Source
Gravitational Constant	G	$6.674\,015\,081 \times 10^{-11}$	Exact	$m^3\ kg^{-1}\ s^{-2}$	Table 2
Electron Mass	m_e	$9.109\,383\,7139 \times 10^{-31}$	11	kg	CODATA
Planck Constant	h	$6.626\,070\,15 \times 10^{-34}$	Exact	$J\ Hz^{-1}$	CODATA
Light Speed	c	$2.997\,924\,58 \times 10^{8}$	Exact	$m\ s^{-1}$	CODATA
Reduced Avogadro Number	$\aleph_A$	$5.989\,020\,203 \times 10^{22}$	Exact	—	Table 4
Grav. Coupling Constant	α_g	$2.787\,972\,231 \times 10^{-46}$	10	—	$Gm_e^2/(hc)$
Grav. Coupling Constant	α_g	$2.787\,972\,231 \times 10^{-46}$	Exact	—	$1/(\aleph_A)^2$

* We have always thought that α_g has a quantum mechanical origin because it utilizes Planck's h; now, nature reveals that the fundamental definition of α_g is the one involving the reduced Avogadro number which, in turn, reveals that the inverse Avogadro factor $f_A^{-1} \simeq 0.10$ (Table 4) is nature's molar unit: thus, $f_A^{-1} \simeq 0.10$ moles of electrons contain $\aleph_A \simeq 6.0 \times 10^{22}$ particles of total mass $M_P \simeq 5.5 \times 10^{-8}$ kg. As a result, a fair number of currently espoused ideas ought to be allowed to fade[5].

2.7. *Table 7*

Table 7 shows the calculation of the relative gravitational coupling ratio $\beta_g \equiv \alpha_g/\alpha$. We have chosen the FSC as the normalizing constant because it is measured to very high precision. Such relative ratios of dimensionless constants can be incorporated into systems of units in the same fashion as the dimensional units. The reference value of the h-defined $\alpha^{-1} \simeq 861$ [6] is also important in resolving the long-standing scientific obsession with the $\hbar$-defined value of number 137 [53,54] which turns out to be irrelevant and unphysical.

Table 7. Gravitational coupling constant α_g from Table 6 and inverse FSC are used to determine the relative coupling ratio β_g shown at the bottom.

Constant	Symbol	Value	SDs	SI Unit	Source
Grav. Coupling Constant	α_g	$2.787\,972\,231 \times 10^{-46}$	Exact	—	Table 6
Inverse FSC	α^{-1}	$861.022\,576\,584$	12	—	CODATA *
Inverse FSC	α^{-1}	$861.022\,576\,6$	10	—	$G_\star(M_P/e)^2$
Relative Grav. Coupling Ratio	β_g	$2.400\,507\,034 \times 10^{-43}$	10	—	α_g/α

* The $\hbar$-dependent CODATA value of $\alpha^{-1} = 137.035\,999\,177(21)$ is multiplied by 2π in accordance with our definition of $\alpha \equiv Ke^2/(hc)$.

2.8. Table 8

Table 8 shows the values of the weak, strong, and gravitational couplings when these constants are expressed in terms of the fundamental constants $\aleph_A$, f_A, and the FSC. These results are new. Comparisons with experimental values, as well as several alternative representations, are given in the notes to the table. We note, in particular, the surprising relations $\alpha_w = \sqrt{\alpha}$ and $\alpha_w = e/Q_P$.

Table 8. Inverse FSC from Table 7 and Avogadro constants from Table 4 are used to determine the fundamental coupling constants shown at the bottom. Copied from Table 6, constant α_g completes the list of couplings *.

Constant	Symbol	Value	SDs	SI Unit	Source
Inverse FSC	α^{-1}	861.022 576 584	12	—	Table 7
Avogadro Factor	f_A	10.0553 0213	Exact	—	Table 4
Reduced Avogadro Number	$\aleph_A$	$5.989\,020\,203 \times 10^{22}$	Exact	—	Table 4
Fine-Structure Constant	α	$1.161\,409\,7321 \times 10^{-3}$	11	—	$1/\alpha^{-1}$
Weak Coupling Constant	α_w	$3.407\,946\,203 \times 10^{-2}$ **	10	—	$\sqrt{\alpha}$
Strong Coupling Constant	α_s	$1.174\,290\,938 \times 10^{-1}$ ***	10	—	$(f_A)^2\,\alpha$
Grav. Coupling Constant	α_g	$2.787\,972\,231 \times 10^{-46}$	Exact	—	$1/(\aleph_A)^2$, Table 6

* Alternative relations involving M_P: $\alpha_g = (m_e/M_P)^2$; $\alpha = (M_S/M_P)^2$, where M_S is the Stoney mass; $\alpha_w = M_S/M_P$; $\alpha_s = (f_A M_S/M_P)^2$. Also, $G_\star = (Q_P/M_P)^2$ and $\alpha_w = e/Q_P$, where Q_P is the Planck charge from Table 3. ** To be compared to the PDG [31] world average of $3.3914(11) \times 10^{-2}$ and the latest CDF [55] value of $3.3969(08) \times 10^{-2}$. *** To be compared to the PDG [31] world average of $1.180(9) \times 10^{-1}$.

2.9. Table 9

Table 9 shows the mass of the W boson calculated from the reduced Fermi constant and the FSC. This result is new. Comparisons with experimental values are given in the notes to the table. The calculated value differs from the latest high-precision CDF measurement [55] by only 0.163%.

Table 9. FSC and reduced Fermi constant are used to determine the mass of the W boson m_W shown at the bottom.

Constant	Symbol	Value	SDs	SI Unit	Source
Fine-Structure Constant	α	$1.161\,409\,732\,1(2) \times 10^{-3}$	11	—	Table 8
Reduced Fermi Constant	G_F^0	$1.166\,378\,8(6) \times 10^{-5}$	8	$(\text{GeV})^{-2}$	PDG [31]
Mass of W boson	m_W	80.5645 5(2) *	7	$(\text{GeV})/c^2$	$(\pi/G_F^0)^{1/2}(\alpha/2)^{1/4}/c^2$

* To be compared to the PDG [31] world average of 80.3692(133) and the latest CDF [55] value of 80.4335(94), both in the same units.

2.10. Table 10

Table 10 shows the complete list of calculated relative coupling ratios $\{\beta_w, \beta_s, \beta_g\}$. These ratios are available to be included in any chosen system of units. We note, in particular, the fundamental relations $\beta_s = (f_A)^2$ and $\beta_w = 1/\sqrt{\alpha}$. These relative coupling ratios are included automatically in any system of measurements with base units f_A, N_A, and α.

Table 10. Avogadro factor and FSC are used to determine the relative weak and strong coupling ratios β_w and β_s shown at the bottom. Copied from Table 7, constant β_g completes the list of relative couplings that are available for use in any chosen system of units.

Constant	Symbol	Value	SDs	SI Unit	Source
Avogadro Factor	f_A	10.0553 0213	Exact	—	Table 4
Fine-Structure Constant	α	$1.161\,409\,7321 \times 10^{-3}$	11	—	Table 8
Relative Strong Coupling Ratio	β_s	101.109 1009 *	Exact	—	$(f_A)^2$
Relative Weak Coupling Ratio	β_w	29.343 186 20	10	—	$1/\sqrt{\alpha}$
Relative Grav. Coupling Ratio	β_g	$2.400\,507\,034 \times 10^{-43}$	10	—	α_g/α, Table 7

* To be compared to $\alpha_s/\alpha = 101.6(8)$ from experiments, where the PDG [31] world average of α_s is 0.1180(9).

2.11. Table 11

Table 11 summarizes the determinations of the Stoney units of mass and length [28] calculated from the two G-Ms of the constants e^2 and $G_\star$. The Stoney length L_S holds a surprise: the leading coefficient of its numerical value (1.380 649) is identical to that of Boltzmann's constant k_B, yielding an independent determination of this unit from the exact equality $\mathcal{N}(L_S) = \mathcal{N}(k_B) \times 10^{-13}$.

The factor of 10^{-13} is composite, formed as the product of 10^{-7} carried by $\mu_0/(4\pi)$ and 10^{-6} contained in the unit of $k_{B,MJ}$ (last note in Table 11). This (sub)multiple may turn out to be common in unit conversions and numerical evaluations. As an example, consider the exact equality

$$\alpha_w = \frac{\mathcal{N}(F_P)}{\mathcal{N}(\Theta_P)} \times 10^{-13}, \tag{9}$$

where $F_P = c^4/G$ and $\Theta_P = M_P c^2/k_B$ are the Planck units of force and temperature, respectively, and $\alpha_w = \sqrt{\alpha}$ in terms of the FSC. The origin of the factor of 10^{-13} is not obvious, as there is no implicit dependence of the units on the magnetic permeability $\mu_0/(4\pi)$. The puzzle becomes easier to solve when one considers the relations $\alpha_w = L_S/L_P$ and $\mathcal{N}(L_S) \propto 10^{-7}$ presented in the notes to Table 11.

Table 11. Stoney units of mass M_S and length L_S, where L_S also represents the electron's charge radius R_e that appears in black-hole physics. These units are effectively expressed by the two G-Ms of e^2 and $G_\star$ *.

Constant	Symbol	Value	SDs	SI Unit	Source
Vacuum Permeability	μ_0	$4\pi \times 10^{-7}$	10	$\mathrm{N\,A^{-2}}$	$1/(\varepsilon_0 c^2)$
Elementary Charge **	e	$1.602\,176\,634 \times 10^{-19}$	Exact	C	CODATA
Effective Grav. Constant	$G_\star$	$7.425\,843\,255 \times 10^{-21}$	Exact	$\mathrm{C^2\,kg^{-2}}$	Table 2
Stoney Mass	M_S	$1.859\,248\,778 \times 10^{-9}$	Exact	kg	$\sqrt{e^2 (G_\star)^{-1}}$
Stoney Length	L_S or R_e	$1.380\,649\,000 \times 10^{-36}$ ***	10	m	$\left(\frac{\mu_0}{4\pi}\right)\sqrt{e^2 G_\star}$

* These units can also be expressed in terms of Planck units and coupling constants: $M_S = M_P \alpha_w$ and $L_S = L_P \alpha_w$, where $\alpha_w = \sqrt{\alpha}$ as well. ** Also adopted as the Stoney charge. *** Since $(\frac{\mu_0}{4\pi})$ introduces only a power of 10^{-7} to the source, then $\mathcal{N}(L_S) = \mathcal{N}(k_{B,MJ}) \times 10^{-7}$.

3. Classical Determinations of QED Constants

Based on the calculations summarized in Tables 1–11, we reformulate some of the important constants of quantum theory in classical terms. Our starting point is the reduced Avogadro number $\aleph_A \simeq 6 \times 10^{22}$ and the associated Avogadro factor $f_A \simeq 10$, the two new natural constants determined more precisely in Table 4. All by themselves, these fundamental constants yield simple expressions for the gravitational coupling $\alpha_g \equiv Gm_e^2/(hc)$ and the relative strong coupling $\beta_s \equiv \alpha_s/\alpha$, respectively, viz.

$$\alpha_g = (\aleph_A)^{-2}, \tag{10}$$

and

$$\beta_s = (f_A)^2. \tag{11}$$

Equation (11) is consistent with the fact that the strong coupling is stronger than the EM FSC coupling since $f_A \gg 1$ and $(f_A)^2$ is even larger. On the other hand, Equation (10) supports the well-known view that gravity's coupling is extremely weak because there exist way too many particles in the universe. Equivalently, since

$$\alpha_g = \left(\frac{m_e}{M_P}\right)^2, \tag{12}$$

gravity's coupling is extremely weak because the mass of the electron is much smaller than the Planck mass.

Next, by combining Equations (10) and (12), we cast the Planck mass in the convenient form

$$M_P = \aleph_A m_e, \tag{13}$$

that will allow us to rewrite many of the equations of physics and all the Planck units in simple forms admitting straightforward interpretations. We reformulate thus some natural constants in the following subsections, a number of well-known QED equations in Section 4, and the Planck system of units in Section 5.

3.1. Planck's Constant h

Solving our h-defined equation for the Planck mass ($M_P = \sqrt{h\,c/G}$) for h, we find that

$$h = \frac{GM_P{}^2}{c}, \tag{14}$$

where M_P is now defined by Equation (13).

This form is not unexpected. It has not been discussed previously because there was no way to derive M_P without using h. Nevertheless, Equation (14) is not the simplest form that can be obtained now. In Section 5, we obtain more forms, the simplest of which turns out to be

$$h = Q_P{}^2\mathcal{R}_P, \tag{15}$$

where

$$Q_P = M_P\sqrt{G_\star}, \tag{16}$$

is the Planck charge. Evidently, Planck's h can be interpreted as a purely EM constant imprinted by the impedance of free space (Equation (2)). This opens a new way of thinking about photons that have never before been thought to be subject to constraints set by the vacuum.

Equations (14) and (15) also lend support to the idea that there exists a unified conservative field responsible for both gravitational and Coulomb long-range forces—because different subsets of the field properties produce consistently the same h value. Naturally, it is Equation (16) that underwrites this consistent behavior.

3.2. Fine-Structure Constant α

The definition of the FSC $\alpha \equiv Ke^2/(hc)$ is recast in the classical form

$$\alpha \;=\; \frac{e^2}{G_\star {M_\mathrm{P}}^2}\,. \tag{17}$$

where M_P is now defined by Equation (13).

Since $G_\star {M_\mathrm{P}}^2 = {Q_\mathrm{P}}^2$ (Equation (16)), we rewrite Equation (17) in the simplest possible form

$$\alpha \;=\; \left(\frac{e}{Q_\mathrm{P}}\right)^2. \tag{18}$$

The correspondence between Equations (12) and (18) is undeniable. We note, additionally, that the same ratios without the squares also have clear physical meanings, viz. $\frac{m_\mathrm{e}}{M_\mathrm{P}} = \left(\aleph_\mathrm{A}\right)^{-1}$ and $\frac{e}{Q_\mathrm{P}} = \alpha_\mathrm{w}$, respectively.

3.3. Relative Gravitational Coupling Constant β_g

The relative gravitational coupling constant $\beta_\mathrm{g} \equiv \alpha_\mathrm{g}/\alpha$ [6] measures the relative strength of gravitational coupling against the measurable by experiment FSC. It is interesting that β_g is independent of the relative Avogadro number $\aleph_\mathrm{A}$ because the α-couplings in its definition are both $\propto (M_\mathrm{P})^{-2}$. This is also seen in the equivalent expression in terms of the mass-to-charge ratio of the electron, viz.

$$\beta_\mathrm{g} \;=\; G_\star \left(\frac{m_\mathrm{e}}{e}\right)^2. \tag{19}$$

The relative ratio β_g is a minimum since $G_\star \propto \varepsilon_0$ and ε_0 is a universal lower limit.

3.4. Compton Radius r_c of the Electron

Eliminating h from the h-defined Compton radius of the electron $r_\mathrm{c} = h/(m_\mathrm{e}c)$, we find that

$$r_\mathrm{c} \;=\; \frac{G{M_\mathrm{P}}^2}{m_\mathrm{e}c^2}\,. \tag{20}$$

This relation admits the classical interpretation that the gravitational binding energy of two Planck masses separated by distance r_c is equal to the rest-energy of one electron[6].

3.5. Landé g_s-Factor of the Electron

Our rejection of $\hbar$ in favor of Planck's h finds additional support from a well-known QED result, the "unambiguous and unambiguously correct determination" [56] of the first-order correction to the Landé g_s-factor of the anomalous magnetic moment of the electron [57,58], viz.

$$\frac{g_\mathrm{s}-2}{2} \;=\; (861.0225766)^{-1} \;=\; 1\alpha\,. \tag{21}$$

The calculation produced a pure numerical value of $\mathcal{O}(\alpha) = 1\alpha$ (where α is defined here in terms of h as in Table 7), but it was not recognized as such (e.g., [56]) because of the $\hbar$-defined FSC at that time. So, the erroneous geometric imprint of $1/(2\pi)$ became the main result, the coefficient in the first-order correction that Schwinger [58] set out to determine by perturbation theory.

No-one noticed the suspicious appearance of the 2-D (2π) term in this result: the magnetic moment and the spin of the electron are vectors, thus the Landé g_s-coefficient should have been a pure number, a scaling constant devoid of geometry, just like the zeroth-order factor $(g_\mathrm{s})_0 = 2$ [59]. Thus, a reasonable interpretation of the result would have been the following:

- Assuming that the calculation was correct, the (2π) tag could not be eliminated by any means; but it could be absorbed in the FSC (ringing the bell that something was not set properly in the definition of that man-made constant at that time). That would have restored the FSC to the self-consistent form given in Table 7, and the correction to the Landé g_s-factor to the pure value of 1α.

4. Geometrically Clear QED Equations

Planck units not using $\hbar$, with Planck's h given by Equation (14) and the Planck mass M_P defined by Equation (13), simplify a large number of physical quantities and allow for unequivocal interpretations of the resulting equations. We summarize here five cases of general interest:

① The Bekenstein-Hawking formula for the entropy of a black hole of mass M_{BH} [60–62] is $S_{BH} = k_B A/(2L_P)^2$, where A is the area of its event horizon and L_P is the Planck length [20]. For a Schwarzschild black hole, we set its horizon area to $A = 4\pi {R_S}^2$, and we also define the Planck length in terms of h, not $\hbar$ (Table 3); then, the Bekenstein-Hawking formula takes the concise form

$$S_{BH} = 4\pi k_B \left(\frac{M_{BH}}{M_P}\right)^2 . \tag{22}$$

The factor of 4π (the imprint of the 3-D space [6,32]) has emerged in this equation to denote that S_{BH} is the integrated entropy enclosed within the volume of the black hole. For a black hole with mass $M_{BH} = m_e$, Equation (22) reduces to $S_{BH} = 4\pi k_B \alpha_g$, where $\alpha_g = (\aleph_A)^{-2}$.

② The Bekenstein bound for the maximum entropy of a body of mass M, radius R, and rest-energy E [63–67] is $S_{max} = k_B(2\pi R)E/(hc)$. Written in this form, the equation gives a misleading signal (i.e., the circumference $(2\pi R)$ is a 2-D quantity), although it reduces to Equation (22) for a black hole with $R = R_S$ and $E = c^2 M_{BH}$. The apparent geometric issue is resolved when S_{max} is reformulated in terms of the Planck mass: using Equation (14) to eliminate (hc) from S_{max}, we find that

$$S_{max} = 4\pi k_B \left(\frac{R}{R_S}\right)\left(\frac{M}{M_P}\right)\left(\frac{E}{E_P}\right), \tag{23}$$

where $E_P = c^2 M_P$ is the Planck energy. The appearance of the comparative ratio R/R_S asserts the fundamental nature of the Schwarzschild radius R_S [65–67] (in contrast to the man-made Planck length L_P), including the natural (i.e., not man-made) factor of 2 that appears in the definition $R_S \equiv 2GM/c^2$: introducing the ratio R/L_P in Equation (23) leads to a simpler formula, viz.

$$S_{max} = 2\pi k_B \left(\frac{R}{L_P}\right)\left(\frac{E}{E_P}\right),$$

which, however, displays the apparent 2π geometric issue previously discussed, arising from the subjective definition of the Planck length.

③ The thermal Hawking temperature of a black hole (also called Hawking-Unruh or Davies-Unruh temperature in related contexts) [68–71] is defined here as $\Theta_{BH} = ha/(k_B c)$, where a denotes acceleration. As usual, this definition is given in terms of h (not $\hbar$), but it is also devoid of a man-made factor[7] of $(2\pi)^2$. For a

Schwarzschild black hole of mass $M_{\rm BH}$ and surface acceleration of $a = GM_{\rm BH}/R_S{}^2 = c^4/(4GM_{\rm BH})$ on the horizon, we find a concise formula for $\Theta_{\rm BH}$, viz.

$$\frac{\Theta_{\rm BH}}{\Theta_{\rm P}} = \frac{1}{4}\left(\frac{M_{\rm BH}}{M_{\rm P}}\right)^{-1}, \tag{24}$$

where $\Theta_{\rm P} = c^2 M_{\rm P}/k_{\rm B}$ is the Planck temperature. The factor of $1/4$ stems from the maximum relativistic tension force [72–75], viz.

$$F_{\rm max} = \frac{c^4}{4G}, \tag{25}$$

which is realized on the horizon $R = R_S$ of the Schwarzschild black hole, where the acceleration is $a = F_{\rm max}/M_{\rm BH}$.

④ A new deeper interpretation of Heisenberg's position-momentum (x, p_x) uncertainty principle [76–78] emerges from Equations (14) and (15):

- Written in the standard form $\Delta x\, \Delta p_x \geq \hbar/2$, the inequality is misleading: Dirac's $\hbar$ is a 2-D constant, whereas the standard deviations $(\Delta x, \Delta p_x)$ are 1-D uncertainties. This recurring issue with $\hbar$ was exposed and explored in Ref. [6] for the first time. Thus, we write the uncertainty principle in an unambiguous form as

$$\Delta x\, \Delta p_x \geq \frac{h}{4\pi}, \tag{26}$$

  that shows a 3-D vacuum tag of 4π, a signature that the 1-D motion actually unfolds within the 3-D space.
- Although the geometry in Equation (26) is now clear, there is another issue that has not heretofore been discussed: Planck's h has been introduced as a lower limit without justification or explanation of its minimum value. In fact, up until now, h has been thought as a constant threshold; perhaps like the vacuum impedance Z_0 and MOND's critical acceleration a_0, and certainly unlike the limiting values c, ε_0, and μ_0.
- This issue is resolved by considering either one of Equations (14) and (15). The lower bound in Heisenberg's inequality, viz. $h/(4\pi)$, is then understood in two fully consistent ways:
  (a) Equation (14) shows that $h \propto 1/c$, hence h attains a minimum value in the natural world because c is an upper limit.
  (b) Equation (15) points to the same conclusion. The Planck resistance is a threshold to be matched from above or below for efficient radiation transmission (see, e.g., [79]). But then, $h \propto Q_{\rm P}{}^2 \propto G_\star \propto \varepsilon_0$, hence h is minimized by the vacuum[8].

It is interesting to note that, in contrast to Planck's h and gravity's $\alpha_{\rm g}$, the dimensionless couplings α, $\alpha_{\rm w}$, and $\alpha_{\rm s}$ attain maximum values. In the ordered list $\alpha_{\rm g} < \alpha < \alpha_{\rm w} < \alpha_{\rm s}$, the reduced Avogadro number and its factor set the extreme values at the two ends, and the vacuum enhances the values of the electroweak constants in the middle[9].

⑤ The Casimir force per unit area between two parallel conducting plates [80] has occupied many physicists over the past 80 years. Its magnitude was determined by several different methods (e.g., [80–84]), and it was confirmed experimentally to $\lesssim$1% accuracy (e.g., [85–87], and references therein). The Casimir effect was originally thought to be a quantum effect that originates from vacuum energy fluctuations and provides proof that zero-point energies in quantum-field ground states are real. These notions were conclusively refuted [84,88,89], except for the quantum nature of the effect ($\hbar$ is present in the equations). In our times, the Casimir force is believed to be the relativistic analogue of the classical van der Waals force in which retardation effects are taken into account [84,88–93], and it is produced by the matter-EM interaction term in the QED Hamiltonian [88].

Here, we revisit the Casimir effect in light of our results:

- Equations (13)–(15) highlight the classical origin of Planck's h, thereby dispelling the notion that the nature of the Casimir force lies in quantum mechanics. Thus, this force is the classical van der Waals force [90,94] corrected to account for the finite speed of light.
- Another issue concerns the appearance of geometric terms in the equations for the Casimir effect. The full treatment of the effect shows 2π-dependent coefficients introduced by counting the density of states along the principal directions on the surfaces of the plates, which does not raise any concerns. Expressed in terms of the Planck unit of pressure $P_\mathrm{P} = c^2 M_\mathrm{P}/{L_\mathrm{P}}^3$, the Casimir pressure P_c is given by

$$P_\mathrm{c} \;=\; -\,\frac{\pi}{480}\,\frac{G{M_\mathrm{P}}^2}{d^4} \;=\; -\,\frac{\pi}{480}\,P_\mathrm{P}\left(\frac{L_\mathrm{P}}{d}\right)^4\,, \tag{27}$$

  where d is the distance between the flat, parallel, perfectly conducting plates. The final π term effectively arises from the quotient of the density of states $\propto (2\pi)^2$ to the area $\propto \pi$. No other factors of π appear in the integration over d to find the binding energy of the plates.
- On the other hand, the simplified 1-D scalar analogue of the effect [83,84] should not contain any geometric terms, which is indeed the case for the 1-D Casimir force F_c, viz.

$$F_\mathrm{c} \;=\; -\,\frac{1}{48}\,\frac{G{M_\mathrm{P}}^2}{d^2} \;=\; -\,\frac{F_\mathrm{max}}{12}\left(\frac{L_\mathrm{P}}{d}\right)^2\,, \tag{28}$$

  where $F_\mathrm{max} = F_\mathrm{P}/4$ is given by Equation (25) above.

5. Reformulated Planck System of Units

Equations (13)–(15) support a reformulation of the conventional Planck system of units [1,2,20] in terms of the fundamental RPS set of universal (field+vacuum+molar) constants

$$\mathrm{RPS} \;:=\; \{\,e,\; m_\mathrm{e},\; k_\mathrm{B};\;\; \varepsilon_0,\; \mu_0;\;\; N_\mathrm{A},\; f_\mathrm{A}\,\}\,. \tag{29}$$

Of those, only the Avogadro factor ($f_\mathrm{A} = 10.0553\,0213^{10}$; Table 4) is an unfamiliar constant. All other natural constants, including G (Table 2), h (Equations (13)–(15)), and the dimensionless couplings (Table 8), can be derived from this fundamental set, starting with $\aleph_\mathrm{A} = N_\mathrm{A}/f_\mathrm{A}$, $M_\mathrm{P} = \aleph_\mathrm{A} m_\mathrm{e}$, and following with the other equations given in Section 3 above.

The original (mechanical) Planck units are listed in Table 12, and the extended (EM) units are listed in Table 13, along with the corresponding RPS units shown in the last column of each table. Mechanical RPS units in Table 12 are given in terms of a subset of units that includes Boltzmann's k_B and the derivatives $\{M_\mathrm{P},\, G,\, c\}$. EM RPS units in Table 13 are given in terms of a subset of units that includes the derivatives $\{M_\mathrm{P},\, G_\star,\, G_B,\, c,\, Z_0\}$. As seen in Tables 1 and 2, Boltzmann's constant and the gravitational constants carry entropy information where they appear.

Table 12. Mechanical Planck units reformulated in terms of the subset of units $\{M_P, G, c, k_B\}$ of which only k_B is a fundamental constant *.

Unit	Symbol	Planck Definition	Reformulation
Mass	M_P	$M_P = \sqrt{h\,c/G}$	$M_P = \aleph_A\, m_e$
Length	L_P	$L_P = \sqrt{h\,G/c^3}$	$L_P = M_P\, G/c^2$
Time	T_P	$T_P = \sqrt{h\,G/c^5}$	$T_P = M_P\, G/c^3$
Temperature	Θ_P	$\Theta_P = \sqrt{h\,c^5/G}/k_B$	$\Theta_P = M_P\, c^2/k_B$
Force	F_P	$F_P = c^4/G$	$F_P = M_P\, c^2/L_P$
Pressure	P_P	$P_P = c^7/(h\,G^2)$	$P_P = M_P\, c^2/{L_P}^3$
Acceleration	a_P	$a_P = \sqrt{c^7/(h\,G)}$	$a_P = c^2/L_P$

* Our RPS set of fundamental units includes field+vacuum+molar constants, viz. $\{e,\ m_e,\ k_B;\ \ \varepsilon_0,\ \mu_0;\ \ N_A,\ f_A\}$. All other constants are derived from them.

Table 13. EM Planck units reformulated in terms of the subset of derivative units $\{M_P, G, c, Z_0\}$ *. The EM units are simplified considerably by the introduction of the effective gravitational constants $G_\star = G/K = 4\pi\varepsilon_0\, G$ (electric) and $G_B = GK/c^2 = G\,\mu_0/(4\pi)$ (magnetic).

Unit	Symbol	Planck Definition	Reformulation **
Charge	Q_P	$Q_P = \sqrt{h\,c/K}$	$Q_P = M_P\sqrt{G_\star}$
Magnetic Flux	Φ_P	$\Phi_P = \sqrt{K\,h/c}$	$\Phi_P = M_P\sqrt{G_B}$
Voltage	$\mathcal{V}_P$	$\mathcal{V}_P = \sqrt{K\,c^4/G}$	$\mathcal{V}_P = c^2/\sqrt{G_\star}$
Electric Current	$\mathcal{I}_P$	$\mathcal{I}_P = \sqrt{c^6/(GK)}$	$\mathcal{I}_P = c^2/\sqrt{G_B}$
Electric Resistance	$\mathcal{R}_P$	$\mathcal{R}_P = K/c$	$\mathcal{R}_P = Z_0/(4\pi)$
Capacitance	C_P	$C_P = \sqrt{h\,G/(K^2\,c^3)}$	$C_P = G_\star\, M_P/c^2$ ***
Inductance	$\mathcal{L}_P$	$\mathcal{L}_P = \sqrt{h\,G\,K^2/c^7}$	$\mathcal{L}_P = G_B\, M_P/c^2$ ***

* Our RPS set of fundamental units includes field+vacuum+molar constants, viz. $\{e,\ m_e,\ k_B;\ \ \varepsilon_0,\ \mu_0;\ \ N_A,\ f_A\}$. All other constants are derived from them. ** The reformulated units produce simple classical expressions for Planck's constant h: (1) $h = G{M_P}^2/c$, showing that h has dimensions of action; (2) $h = G_\star {M_P}^2 \mathcal{R}_P$, where the interplay between gravity and the vacuum constants in formulating the classical form of h becomes apparent; (3) the definition of the Planck charge implies the fundamental relation ${Q_P}^2 = G_\star {M_P}^2$ which leads to $h = {Q_P}^2 \mathcal{R}_P$, where the interplay between EM forces and the vacuum becomes apparent. In a unified conservative field that contains two sources (mass $G_\star M$ and charge Q), such determinations leading to a unique value of h are not surprising. *** Notably also $C_P = (4\pi\varepsilon_0)L_P$ and $\mathcal{L}_P = \left(\frac{\mu_0}{4\pi}\right)L_P$.

The effective gravitational constants $G_\star$ and G_B appear in the units of Table 13 in a systematic way: $G_\star$ appears in electrostatic units, whereas G_B appears in magnetic units associated with current flows. This distinction is also seen in the SI units of the two constants:

- The SI unit of $\sqrt{G_\star}$ is $\mathrm{C\,kg^{-1}}$, hence this constant represents a charge-to-mass ratio. Thus, $\sqrt{G_\star}$ could be an integral part of a unified conservative long-range field that would combine the sources of mass and charge.
- The SI unit of $\sqrt{G_B}$ is $(\mathrm{m^2\,s^{-1}})\,\mathrm{C^{-1}}$, hence this constant represents areal flux per unit charge.

The above constants are composite, so their squares roots represent G-Ms. Combined, they also form two new G-Ms. The roles of the various G-M derivatives are elucidated below.

5.1. The Geometric Means $\sqrt{G_\star}$ and $\sqrt{G_B}$

(a) The composite constant $G_\star$ (Equation (4)) appeared in an effort to bring Gauss's law into precise correspondence between the gravitational field and the electrostatic field [32], but its G-M $\sqrt{G_\star}$ gave us the motivation to pursue the present research by revealing a numerical connection between this G-M and k_B/e (Tables 1 and 2), viz.

$$\mathcal{N}(\sqrt{G_\star}) \;=\; \frac{\mathcal{N}(k_B)}{\mathcal{N}(e)} \times 10^{-6}\,, \tag{30}$$

followed by the additional relations pertaining to MOND constants that, for all practical purposes, show that $\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(G_\star)$ and $\mathcal{N}(a_0) = \mathcal{N}(4\pi\varepsilon_0)$ (Table 5).

(b) The composite constant G_B (Equation (5)) appeared in the reformulation of the EM Planck units (Table 13), where it simplified greatly the new RPS units. Its G-M $\sqrt{G_B}$ carries precisely the same information as $\sqrt{G_\star}$, although it is scaled by a different vacuum constant, viz.

$$\sqrt{G_B} \;=\; \mathcal{R}_{\rm P}\sqrt{G_\star} \;=\; \frac{Z_0}{4\pi}\sqrt{G_\star}\,. \tag{31}$$

5.2. The Geometric Means of $G_\star$ and G_B

(a) The G-M of G_B and $1/G_\star$ is obtained from Equation (31), viz.

$$\sqrt{\frac{G_B}{G_\star}} \;=\; \mathcal{R}_{\rm P}\,. \tag{32}$$

In the form of the Planck resistance $\mathcal{R}_{\rm P}$, this G-M is multitasking in EM relations, as it appears in the units of Ohm's law $\mathcal{V}_{\rm P}/\mathcal{I}_{\rm P} = \mathcal{R}_{\rm P}$, the units of the magnetic flux $\Phi_{\rm P}/Q_{\rm P} = \mathcal{R}_{\rm P}$ in Faraday's law, and the RLC circuit units of $\mathcal{L}_{\rm P}/\mathcal{C}_{\rm P} = \mathcal{R}_{\rm P}{}^2$ (Table 13). Furthermore, it is easy to show that $\sqrt{G_B/G_\star} = K/c$, where $K = 1/(4\pi\varepsilon_0)$ is Coulomb's constant. This relation is useful for comparison with the other G-M discussed just below.

(b) The other G-M obtained from the product of G_B and $G_\star$ is certainly related to Newton's G. A simple reduction yields the relation $\sqrt{G_B\,G_\star} = G/c$.

Finally, combining the two G-Ms, we obtain the unfamiliar magnetic relation mentioned in Table 13 that $G_B = GK/c^2$. From this form, we can also derive the G-M of G and K, viz.

$$\sqrt{GK} \;=\; c\sqrt{G_B}\,, \tag{33}$$

also unfamiliar in physical theory. This G-M could provide the constant of interaction between mass and charge in a unified conservative long-range field [95].

6. Discussion and Conclusions

6.1. Results

In Section 1, we categorized and summarized several conceptual pitfalls affecting physical theory, as well as the substantial benefits that may follow from their resolution. The key empirical findings of this investigation are presented in Tables 1–13 and may be grouped into the following seven categories:

1. Planck's constant is certainly more important than previously thought:

- ➢ Dirac's $\hbar$ should be recognized for what it is (a 2-D composite constant appropriate for planar orbits and 2-D spaces), and it should not be used in 3-D settings such as man-made coupling constants and Planck units, where Planck's h is the correct constant to be utilized. When this is done, several 'pieces of the puzzle' fall into place, as follows.
- ➢ The h-defined FSC $\alpha \simeq 1/861$ displays number 861 that can be interpreted physically [6], very much unlike the unphysical 137 that has tormented many physicists in the past [53,54].

2. Planck mass and gravitational coupling can be obtained from Avogadro's number:

- ➢ The Planck mass $M_{\rm P} = \aleph_{\rm A} m_{\rm e}$ is related to the electron mass $m_{\rm e}$ via a reduced Avogadro number $\aleph_{\rm A} = N_{\rm A}/f_{\rm A}$ which also determines uniquely the gravitational coupling constant $\alpha_{\rm g} = 1/(\aleph_{\rm A})^2$.
- ➢ The gravitational interaction (quantified via $\alpha_{\rm g}$) is extremely weak in nature because there exist way too many particles ($\aleph_{\rm A} \gg 1$).

- In this formulation, fundamental QED constants such as Planck's h, the FSC, and the Compton radius of the electron are found to be derivatives that have a classical origin (Section 3); whereas classical Avogadro numbers such as $\aleph_A$ and f_A appear to be fundamental natural constants.

3. Couplings of the weak and the strong forces and masses of vector bosons:

- The weak coupling constant turns out to be $\alpha_w = \sqrt{\alpha}$, indicating that electroweak theory actually has only one coupling constant.
- The Avogadro factor f_A effectively determines the strong coupling constant $\alpha_s = (f_A)^2\alpha$. Equivalently, the relative ratio $\beta_s \equiv \alpha_s/\alpha$ is determined solely from f_A, viz. $\beta_s = (f_A)^2 \gg 1$.
- The mass of the W boson is determined to high precision from the reduced Fermi constant and the FSC (Table 9). The mass of the Higgs boson is also found by a G-M in Section 6.2 below.

4. Fundamental units, reformulated Planck units, and their derivatives:

- The RPS set of (field+vacuum+molar) constants $\{e,\ m_e,\ k_B;\ \ \varepsilon_0,\ \mu_0;\ \ N_A,\ f_A\}$ appears to be a fundamental set of units that produces an uncomplicated, easy-to-use reformulation of the original Planck system of units (Tables 12 and 13).
- Using EM RPS units (Table 13), Planck's constant can be written in the simplest possible classical form, viz. $h = Q_P{}^2\mathcal{R}_P$. Hence, Planck's h can be interpreted as a purely EM constant modified only by the impedance of free space (Equation (2)).

5. Effective gravitational constants and gravity's entropic content:

- The value of Newton's gravitational constant G is determined from the values of Boltzmann's constant k_B, the elementary charge e, and the vacuum constant $4\pi\varepsilon_0$ (Table 2).
- Two new effective gravitational constants are defined by combining G with vacuum constants, viz. $G_\star = G(4\pi\varepsilon_0)$ and $G_B = G(\frac{\mu_0}{4\pi})$, that are related by $G_B = \mathcal{R}_P{}^2 G_\star$. They are both minimum values since ε_0 and μ_0 are lower limits in nature.
- All constants determined in terms of Newton's G carry entropy-related embedded information. The numerical coefficient of the exact SI value of Boltzmann's constant (1.380 649 [13,14]) appears explicitly in the values of the G-M $\sqrt{e^2 G_\star}$ and the charge radius of the electron R_e [37] (or, equivalently, the Stoney length L_S [28]; Table 11). This is the first time that entropy considerations have been identified in a linear (1-D) setting.

6. Geometric-mean relations of universal constants:

- Numerous physical constants and two of 4 vacuum constants are determined from G-M averages involving other universal constants [6,35,36]. Several cases have been highlighted in the main text, and the key G-M $\sqrt{e^2 G_\star}$ was just mentioned above. In this regard, the other G-M of e^2 and $G_\star$ (i.e., the Stoney mass M_S) is not well-known and is discussed below in Appendix A.2. A complete list of all related constants is summarized in Appendix A.3.
- The G-M of the Planck charge Q_P and the elementary charge e produces a new intermediate charge $Q_\star$. The resulting geometric sequence $\{e < Q_\star < Q_P\}$ has an unusual common ratio of $\alpha_w{}^{-1/2} \simeq 5.4169$ (since $e/Q_P = \alpha_w$). The significance of charge $Q_\star$ is not clear yet[11].

7. Source of gravity in MOND:

- The source of the gravitational field in Newtonian dynamics and in MOND [32–34,49] has the same strength, viz. $\mathcal{N}(\mathcal{A}_0 M) = \mathcal{N}(G_\star M)$ for a mass M (Appendix A.1). But the

MOND force is modified by an overlaid square root, in which case the units of the effective gravitational constant are modified accordingly, but not its numerical value.

➢ MOND's modification of the source of gravity $\mathcal{A}_0 M$ by a square root [49] is analogous to $\sqrt{\alpha}$ producing the value of α_w (even though $\mathcal{A}_0$ is not unitless). This analogy could not have been made until recently [6,32] for the relation $\alpha_w = \sqrt{\alpha}$ was not known.

➢ For the MOND threshold acceleration a_0, we then find a non-cosmological 'vacuum' value of $\mathcal{N}(a_0) = \mathcal{N}(4\pi\epsilon_0)$, falling well within the error bar of recent astrophysical measurements [96,97].

6.2. Universal Masses

Nature seems to recognize $\aleph_A \simeq 6.0 \times 10^{22}$ as fundamental constant and the inverse Avogadro factor $f_A{}^{-1} \simeq 0.10$ as a universal molar unit (Table 4). Then, the Planck mass $M_P = \aleph_A m_e$ is asserted as an important mass scale (Table 3), and the mass of the electron m_e is recognized as a fundamental natural unit (Section 5).

There are several notable relations involving these masses:

- The mass ratio m_e/M_P is a measure of the strength of gravity (Equation (12)).
- The G-M $\sqrt{m_e M_P}$ is a multiple of the Higgs mass m_H [31], viz. $\sqrt{m_e M_P} = 10^6 m_H$ accurate to within 0.12% (see Appendix A.3 for details).
- The scaling $M_P \alpha_w$ produces the Stoney mass M_S at which the relative gravitational coupling $\beta_g = 1$ (Appendix A.2).
- The scaling m_e/α_w produces a new subatomic mass scale $M_{sub} = 14.9943\,3735$ MeV/c^2 that lies between the masses of the light quarks.
- Mass M_{sub} was first obtained from a scaling of the Planck mass, viz. $M_{sub} = M_P\sqrt{\beta_g}$ [6], where β_g is defined by Equation (19).

The involvement of the weak coupling constant α_w may not be a surprise [6]. It is plausible that $\alpha_w = \sqrt{\alpha}$ also comes from a ratio of masses of elementary particles: using the most recent PDG values [31] for the masses of the bottom quark (m_b) and the Higgs boson (m_H), we find that $m_b/m_H = 3.3411(63) \times 10^{-2}$, lower by 1.5% than the PDG world average of α_w (notes to Table 8). This supports the relation $\alpha = (m_b/m_H)^2$ which is analogous to that of Equation (12), viz. $\alpha_g = (m_e/M_P)^2$.

6.3. Universal Charges

Along similar lines to the masses above, the Planck charge $Q_P = M_P\sqrt{G_\star}$ is asserted as an important charge scale (Table 3), and the elementary charge e is recognized as a fundamental natural unit (Section 5).

There are some notable relations involving these charges as well:

- The charge ratio e/Q_P is a measure of the strength of the electroweak interaction (viz. $(e/Q_P)^2 = \alpha$ and $e/Q_P = \alpha_w$, respectively).
- The G-M $\sqrt{e Q_P}$ is a new charge $Q_\star$ whose significance is not yet known.
- The Planck charge $Q_P \propto \sqrt{\varepsilon_0}$ appears to be a minimized value (ε_0 is a lower limit), although the other two charges lie below this minimized threshold.

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Abbreviations

The following abbreviations are used in this manuscript:

CDF	Collider Detector at Fermilab
CODATA	Committee On Data
EM	ElectroMagnetic
FJ	Faber–Jackson
FSC	Fine-Structure Constant
G-M	Geometric-Mean
MOND	MOdified Newtonian Dynamics
PDG	Particle Data Group
ppm	parts per million
QED	Quantum ElectroDynamics
RLC	Resistance-Inductance-Capacitance (circuit)
RPS	Reformulated Planck System
SDs	Significant Digits
SI	Système International d'unités
TF	Tully–Fisher
1-D, 2-D, etc.	one-dimensional, two-dimensional, etc.

Appendix A

Appendix A.1. MOND Universal Constants

In MOND, as well as in varying-G gravity, a fundamental constant appears besides Newton's G [33–35,49,98–102], and it is the only constant that remains in the so-called deep MOND limit in which the Newtonian force is neglected [33,34]. In the deep MOND limit, $G \to 0$ and the critical acceleration $a_0 \to \infty$, while the product $\mathcal{A}_0 = a_0 G$ remains finite. The dimensions of $\mathcal{A}_0$, viz. $[v]^4/[M]$, are reminiscent of the baryonic Tully–Fisher (TF) [103–105] and Faber–Jackson (FJ) relations [106–108], galactic relations that are naturally explained by these theories of modified dynamics and modified gravity.

Constant $\mathcal{A}_0$ has been previously determined approximately from the measured value of Newton's G and an average critical value of $a_0 = 1.20 \times 10^{-10}$ m s^{-2} obtained from observed spiral galaxy rotation curves. The errors are $\pm 0.24 \times 10^{-10}$ m s^{-2} (systematic) and $\pm 0.02 \times 10^{-10}$ m s^{-2} (random) [96,97]. We, on the other hand, have determined the values of $\mathcal{A}_0$ and a_0 from the numerical values of $\mathcal{N}(G_\star)$ and $\mathcal{N}(4\pi\varepsilon_0)$, respectively (Table 5), with the value of a_0 falling well within the observational error bar.

The numerical concurrence between $G_\star$ and $\mathcal{A}_0$ (at a level of 21 orders of magnitude below unity) is not a coincidence. It occurs because a_0 and $4\pi\varepsilon_0$ have equal magnitudes (apart from units). This results in the same strength of the source of the gravitational field in Newtonian dynamics and in MOND as well, viz.

$$\mathcal{N}(\mathcal{A}_0 M) = \mathcal{N}(G_\star M)\,, \tag{A1}$$

for the same mass M; although the MOND force is also modified by a square root which is responsible for the different units between $\mathcal{A}_0$ and $G_\star$ [32]. Thus, there is no need for an equivalence principle of masses in MOND either [49].

The physical interpretation of $\mathcal{A}_0 = a_0 G$ is as follows [6,34,49]: $\mathcal{A}_0$ is the proportionality constant in the TF and FJ relations [103–108], viz.

$$v^4 \;=\; \mathcal{A}_0 M\,, \tag{A2}$$

where v is speed. This raises the question of interpreting the other universal constant $G_\star = 4\pi\varepsilon_0 G$ in the same context: In the EM Planck system of units (Table 13), there is only one unit defined in terms of c and $G_\star$, the unit of voltage $\mathcal{V}_{\rm P} = c^2/\sqrt{G_\star}$. By dimensional analysis, we thus obtain a "TF/FJ-like relation" for the square of the voltage $\mathcal{V}$, viz.

$$v^4 \;=\; G_\star \mathcal{V}^2\,. \tag{A3}$$

Combining Equations (A2) and (A3), we find in SI units that

$$\mathcal{V} \;=\; \left(\frac{M}{1\,\mathrm{kg}}\right)^{1/2}\mathrm{V}\,. \tag{A4}$$

Although it may prove unfeasible to test this relation in individual galaxies, the scaling works for the universe as a whole in a compelling way: Using $M = c^4/\mathcal{A}_0 = 1.08777 \times 10^{54}$ kg for the mass of the universe in the cosmological system of units $\{c, G, a_0\}$ [6], then Equation (A4) returns the Planck voltage

$$\mathcal{V}_{\rm P} \;=\; \frac{c^2}{\sqrt{G_\star}} \;=\; 1.042\,962\,076 \;\times\; 10^{27}\ \mathrm{V}. \tag{A5}$$

This congruence occurs because of the equality (A1).

Appendix A.2. The Geometric Mean $\sqrt{e^2 G_\star^{-1}}$ and Comments on Physical Numerology

Appendix A.2.1. The Stoney Mass $M_{\rm S}$ and the Gravitational Source of the Electron $G_\star m_{\rm e}$

The mostly neglected G-M of e^2 and $G_\star{}^{-1}$ has dimensions of [mass], and it appears in Table 11 as the Stoney mass [109]. Thus, we have

$$M_{\rm S} \;=\; \sqrt{e^2 G_\star{}^{-1}}\,. \tag{A6}$$

The Stoney units of mass and length can be obtained from the corresponding Planck units by multiplication by $\alpha_{\rm w}$ [25,109], so we also have $M_{\rm S} = M_{\rm P}\alpha_{\rm w}$. Furthermore, the value of $M_{\rm S}$ can be obtained from the numerical equality

$$\mathcal{N}(M_{\rm S}) \;=\; \frac{\mathcal{N}(e)^2}{\mathcal{N}(k_{\rm B})} \;\times\; 10^6\,, \tag{A7}$$

leading to the additional equality

$$\mathcal{N}(G_\star M_{\rm S}) \;=\; \mathcal{N}(k_{\rm B}) \;\times\; 10^{-6}\,, \tag{A8}$$

that offers a clear interpretation of the gravitational source term $G_\star M_{\rm S} \propto k_{\rm B}$, or $G_\star M_{\rm P} \propto k_{\rm B}/\alpha_{\rm w}$, or, for the tangible electron,

$$G_\star m_{\rm e} \;\propto\; \frac{k_{\rm B}}{\alpha_{\rm w}\aleph_{\rm A}}\,. \tag{A9}$$

The gravitational source carries entropy, but it is exceptionally weak because there are way too many particles in the universe, limiting the field to effectively one (microscopic or macroscopic) state; that is, $W = \exp\left[10^{-6}/(\alpha_w \aleph_A)\right] \rightarrow 1^+$ in Boltzmann's entropy, so that $\ln W = 10^{-6}/(\alpha_w \aleph_A)$.

Appendix A.2.2. Physical Interpretations of M_S and the FSC

Equation (A6) implies that the attractive Newtonian force between two Stoney masses separated by distance r has the same magnitude as the repulsive Coulomb force between two electrons or protons at the same distance r, viz. $G{M_S}^2/r^2 = Ke^2/r^2$.

Furthermore, mass M_S has significance for particle physics as well: It was first obtained in Ref. [6] from a different perspective (the unification of coupling constants [110,111]), and it was then expressed in atomic units of $(\mathrm{GeV})/c^2$. The argument was that, if the coupling constants are running at higher energies, then the gravitational coupling constant α_g meets the FSC (i.e., $\beta_g = 1$) at the critical mass $M_S = 1.042\,962\,076 \times 10^{18}\ (\mathrm{GeV})/c^2$ [12]. Thus, the Stoney mass M_S may also be determined from Equation (19) by letting $\beta_g \rightarrow 1$ and $m_e \rightarrow M_S$.

Finally, the Stoney units of mass and length afford another physical interpretation of the FSC: By multiplying $M_S = M_P\alpha_w$ by $L_S = L_P\alpha_w$, we obtain the relation

$$\alpha \;=\; \frac{Ke^2/r_c}{m_e\,c^2}\,, \tag{A10}$$

where the Compton radius r_c of the electron is given by Equation (20). We see then that the FSC represents the ratio of the electrostatic potential energy of two elementary charges separated by distance r_c to the rest-energy of one electron[13].

Appendix A.2.3. Physical Numerology and Force Unification

Precise numerical relations between physical constants that carry different units, such as those described by Equations (A1), (A7), and (A8), cannot probably be categorized to either one of the two classes of numerological formulae specified by I. J. Good [112]. In 1990, Good [112] wrote:

> "When a numerological formula is proposed, then we may ask whether it is *correct.* The notion of *exact correctness* has a clear meaning when the formula is purely mathematical, but otherwise some clarification is required. I think an appropriate definition of *correctness* is that the formula has a good explanation, in a Platonic sense, that is, the explanation could be based on a good theory that is not yet known but 'exists' in the universe of possible reasonable ideas."

The numerical equalities discussed in this work are certainly *exactly correct*, yet they do require physical backing. Our understanding is that nature applies constraints with the same strength (and rebrands constants by taking their square roots) in different physical settings that researchers tend to analyze separately and in isolation from the general realm of the physical sciences.

This is an important assertion: We believe that force unification would not be viable without considering such a wider context across fields of physics and without limiting the number of independent constants at all physical scales. In this respect, the determination of G-M relations between natural constants (Tables 1–11 above and Refs. [32,35,95,109]) and between the 19 free parameters of the Standard Model of particle physics [6] appears to be a step in the right direction. This methodology suggests a new approach to formulating SI and Planck units (such as the mole and the coupling constants of the fundamental forces) [109,113–115], and is expected to help in the construction of a comprehensive

Lagrangian for the unified field which would depend on just a few ad-hoc parameters to describe all particle interactions and the cosmological scales [25,116–118].

Appendix A.3. A Comprehensive List of Related Physical Constants

The physical constants investigated in this work are summarized in Tables A1 and A2. The constants are divided into fundamental and derivatives, respectively. Input values have been obtained from the CODATA [14] and the PDG [31] catalogues. Additional Planck units are listed in Tables 12 and 13 of the main text.

Table A1. List of the seven fundamental physical constants The set is defined in Equation (29). The first six constants were taken from CODATA [14], whereas f_A was determined in this work.

Constant	Symbol	SI Value	SI Unit
		FIELD CONSTANTS	
Elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C
Electron mass	m_e	$9.109\,383\,7139 \times 10^{-31}$	kg
Boltzmann's constant	k_B	$1.380\,649 \times 10^{-23}$	$\mathrm{J\,K^{-1}}$
		VACUUM CONSTANTS	
Vacuum permittivity	ε_0	$8.854\,187\,8188 \times 10^{-12}$	$\mathrm{F\,m^{-1}}$
Vacuum permeability	μ_0	$4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2}}$
		MOLAR CONSTANTS	
Avogadro number	N_A	$6.022\,140\,76 \times 10^{23}$	—
Avogadro factor	f_A	$10.0553\,0213$	—

Table A2. List of derived physical constants. The function $\mathcal{N}(x)$ captures the numerical value of x *.

Constant	Symbol	Equation	SI Value	SI Unit
		VACUUM		
Speed of light	c	$c = 1/\sqrt{\varepsilon_0\mu_0}$	$2.9979\,2458 \times 10^{8}$	$\mathrm{m\,s^{-1}}$
Planck resistance	$\mathcal{R}_P$	$\mathcal{R}_P = Z_0/(4\pi)$	$2.9979\,2458 \times 10^{1}$	Ω
Coulomb's constant	K	$K = 1/(4\pi\varepsilon_0)$	$8.987\,551\,786 \times 10^{9}$	$\mathrm{m^3\,kg\,s^{-2}\,C^{-2}}$
		GRAVITY		
Effective $G_\star$	$G_\star$	$\mathcal{N}(G_\star) = \mathcal{N}(10^{-6}k_B/e)^2$	$7.425\,843\,255 \times 10^{-21}$	$\mathrm{C^2\,kg^{-2}}$
Newton's constant	G	$G = G_\star/(4\pi\varepsilon_0)$	$6.674\,015\,081 \times 10^{-11}$	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Effective G_B	G_B	$G_B = G\mu_0/(4\pi)$	$6.674\,015\,081 \times 10^{-18}$	$\mathrm{m^4\,s^{-2}\,C^{-2}}$
MOND fundamental	$\mathcal{A}_0$	$\mathcal{N}(\mathcal{A}_0) = \mathcal{N}(G_\star)$	$7.425\,843\,255 \times 10^{-21}$	$\mathrm{m^4\,kg^{-1}\,s^{-4}}$
MOND acceleration	a_0	$\mathcal{N}(a_0) = \mathcal{N}(4\pi\varepsilon_0)$	$1.112\,650\,0562 \times 10^{-10}$	$\mathrm{m\,s^{-2}}$
		h-DEFINED PLANCK UNITS		
Planck mass	M_P	$M_P = \sqrt{hc/G}$	$5.455\,628\,31 \times 10^{-8}$	kg
Planck length	L_P	$L_P = \sqrt{hG/c^3}$	$4.051\,264\,07 \times 10^{-35}$	m
Planck charge	Q_P	$Q_P = \sqrt{hc/K}$	$4.701\,296\,73 \times 10^{-18}$	C
		NATURAL NUMBER OF PARTICLES		
Avogadro factor	f_A	$f_A = N_A m_e/M_P$	$10.0553\,0213$	—
Reduced Avogadro Number	$\aleph_A$	$\aleph_A \equiv N_A/f_A = M_P/m_e$	$5.989\,020\,203 \times 10^{22}$	—
Gravitational coupling	α_g	$\alpha_g = (\aleph_A)^{-2}$	$2.787\,972\,231 \times 10^{-46}$	—

Table A2. *Cont.*

Constant	Symbol	Equation	SI Value	SI Unit
		FORCE COUPLINGS		
EM coupling (FSC)	α	$\alpha = Ke^2/(hc)$	$(861.022\,5766)^{-1}$	—
Weak coupling	α_w	$\alpha_\mathrm{w} = \sqrt{\alpha}$	$3.407\,946\,203 \times 10^{-2}$	—
Strong coupling	α_s	$\alpha_\mathrm{s} = (f_\mathrm{A})^2\,\alpha$	$1.174\,290\,938 \times 10^{-1}$	—
		RELATIVE COUPLING RATIOS		
Relative strong coupling	β_s	$\beta_\mathrm{s} \equiv \alpha_\mathrm{s}/\alpha = (f_\mathrm{A})^2$	101.109 1009	—
Relative weak coupling	β_w	$\beta_\mathrm{w} \equiv \alpha_\mathrm{g}/\alpha = 1/\sqrt{\alpha}$	29.3431 8620	—
Relative grav. coupling	β_g	$\beta_\mathrm{g} \equiv \alpha_\mathrm{g}/\alpha$	$2.400\,507\,034 \times 10^{-43}$	—
		ELECTROWEAK RELATIONS		
Stoney length	L_S	$L_\mathrm{S} \equiv \frac{\mu_0}{4\pi}\sqrt{e^2 G_\star} = L_\mathrm{P}\alpha_\mathrm{w}$	$1.380\,649\,000 \times 10^{-36}$	m
Stoney length	L_S	$\mathcal{N}(L_\mathrm{S}) = \mathcal{N}(10^{-13}\,k_\mathrm{B})$	$1.380\,649\,000 \times 10^{-36}$	m
Stoney mass	M_S	$M_\mathrm{S} \equiv \sqrt{e^2 (G_\star)^{-1}} = M_\mathrm{P}\alpha_\mathrm{w}$	$1.859\,248\,778 \times 10^{-9}$	kg
Stoney mass	M_S	$\mathcal{N}(M_\mathrm{S}) = \mathcal{N}(\,e^2/(10^{-6}\,k_\mathrm{B}))$	$1.859\,248\,778 \times 10^{-9}$	kg
Gravitational source	$G_\star M_\mathrm{S}$	$\mathcal{N}(G_\star M_\mathrm{S}) = \mathcal{N}(10^{-6}\,k_\mathrm{B})$	$1.380\,649\,000 \times 10^{-29}$	$\mathrm{C^2\ kg^{-1}}$
W boson mass	m_W	$m_\mathrm{w} = \sqrt{(\pi/G_\mathrm{F}^0)(\alpha/2)^{1/2}}/c^2$	80.564 55	$(\mathrm{GeV})/c^2$
Higgs boson mass	m_H	$m_\mathrm{H} = 10^{-6}\sqrt{m_\mathrm{e} M_\mathrm{P}}$	125.054 089 **	$(\mathrm{GeV})/c^2$
		GEOMETRIC-MEAN RELATIONS		
G-M of M_P and $k_\mathrm{B,MJ}$	$Q_\star$	$\mathcal{N}(Q_\star) = \mathcal{N}\left(\sqrt{M_\mathrm{P}(10^{-6}\,k_\mathrm{B})}\right)$	$8.678\,886\,893 \times 10^{-19}$	C
G-M of Q_P and e	$Q_\star$	$Q_\star = \sqrt{e\,Q_\mathrm{P}}$	$8.678\,886\,893 \times 10^{-19}$	C
G-M of Q_P and $1/e$	$\sqrt{Q_\mathrm{P}/e}$	$\sqrt{Q_\mathrm{P}/e} = (\alpha_\mathrm{w})^{-1/2}$	5.416 935 130	—
G-M of G and K	$\sqrt{GK}$	$\sqrt{GK} = c\sqrt{G_B}$	$7.744\,872\,895 \times 10^{-1}$	$\mathrm{m^3\ s^{-2}\ C^{-1}}$
G-M of G and $1/K$	$\sqrt{G_\star}$	$\mathcal{N}(\sqrt{G_\star}) = \mathcal{N}(10^{-6}\,k_\mathrm{B}/e)$	$8.617\,333\,262 \times 10^{-11}$	$\mathrm{C\ kg^{-1}}$

* The numerical relations are simplified by introducing the unit $k_\mathrm{B,MeV} = (10^{-6}\,k_\mathrm{B}/e)$ MeV $\mathrm{K^{-1}}$ (Table 2), e.g., $\mathcal{N}(M_\mathrm{S}) = \mathcal{N}(e/k_\mathrm{B,MeV})$ and $\mathcal{N}(\sqrt{G_\star}) = \mathcal{N}(k_\mathrm{B,MeV})$. A more compact notation in terms of a 'reduced Boltzmann constant' is presented in auxiliary Table A3 below. ** To be compared to the PDG [31] world average of 125.20(11) (GeV)/c^2[14].

Table A3. The numerical relations in Table A2 that display entropy information explicitly are summarized here for comparison purposes. The equalities are simplified by introducing the 'reduced Boltzmann constant' $\mathcal{K}_\mathrm{B} \equiv 10^{-6}\,k_\mathrm{B}$. The function $\mathcal{N}(x)$ captures the numerical value of x.

Constant	Symbol	Equation	SI Value	SI Unit
Gravitational source	$G_\star M_\mathrm{S}$	$\mathcal{N}(G_\star M_\mathrm{S}) = \mathcal{N}(\mathcal{K}_\mathrm{B})$	$1.380\,649\,000 \times 10^{-29}$	$\mathrm{C^2\ kg^{-1}}$
Stoney length	L_S	$\mathcal{N}(L_\mathrm{S}) = \mathcal{N}(\frac{\mu_0}{4\pi}\mathcal{K}_\mathrm{B})$	$1.380\,649\,000 \times 10^{-36}$	m
G-M of G and $1/K$	$\sqrt{G_\star}$	$\mathcal{N}(\sqrt{G_\star}) = \mathcal{N}(\mathcal{K}_\mathrm{B}/e)$	$8.617\,333\,262 \times 10^{-11}$	$\mathrm{C\ kg^{-1}}$
Effective $G_\star$	$G_\star$	$\mathcal{N}(G_\star) = \mathcal{N}(\mathcal{K}_\mathrm{B}/e)^2$	$7.425\,843\,255 \times 10^{-21}$	$\mathrm{C^2\ kg^{-2}}$
G-M of M_P and $\mathcal{K}_\mathrm{B}$	$Q_\star$	$\mathcal{N}(Q_\star) = \mathcal{N}\left(\sqrt{M_\mathrm{P}\mathcal{K}_\mathrm{B}}\right)$	$8.678\,886\,893 \times 10^{-19}$	C
Stoney mass	M_S	$\mathcal{N}(M_\mathrm{S}) = \mathcal{N}(e^2/\mathcal{K}_\mathrm{B})$	$1.859\,248\,778 \times 10^{-9}$	kg

Numerical values of man-made constants have been determined by using Planck's h instead of Dirac's $\hbar$, and Newton's G precise to 10 SDs (Table 4) has been used throughout.

The numerical relations in Table A2 that show explicit dependencies on entropy ($\sim k_\mathrm{B}$) are summarized in auxiliary Table A3, where the equalities are simplified considerably by using the 'reduced Boltzmann constant' $\mathcal{K}_\mathrm{B}$ such that $\mathcal{K}_\mathrm{B} \equiv 10^{-6}\,k_\mathrm{B}$. It seems as though the frequently occurring scaling factor of 10^{-6} to k_B is dimensionless, as it also appears (in the absence of k_B) in the G-M determination of the Higgs boson mass $m_\mathrm{H} = 10^{-6}\sqrt{m_\mathrm{e} M_\mathrm{P}}$

(Table A2). This prevents the assignment of units to this number that would have converted the numerical equalities in Tables A2 and A3 to actual dimensional equations[15].

We do however note some important points in relation to this unitless factor of 10^{-6}:

- The reduced Boltzmann constant $\mathcal{K}_B = 10^{-6} k_B$ is not exotic in any sense. It would naturally replace Boltzmann's k_B if the (man-made) Kelvin temperature scale were to be stretched by a factor of 10^6 (so that the thermal energy $k_B\Theta$ would remain the same).
- The factor of 10^{-6} in $\mathcal{K}_B$ effectively introduces a lower unit of entropy in the relations of Table A3. Only the Stoney mass gets a boost from the lower value of $\mathcal{K}_B$ because $\mathcal{N}(M_S) \propto \mathcal{N}(\mathcal{K}_B)^{-1}$.
- The three important masses in physics form the geometric progression $\{m_e,\ 10^6 m_H,\ M_P\}$ with common ratio $\sqrt{\aleph_A} \simeq 2.44725 \times 10^{11}$. This ratio that describes a concise relation between the three masses effectively depends on the number of particles in the present universe.
- The G-M $10^6 m_H$ describes the cumulative mass of one million Higgs bosons, so the mass of the Higgs boson m_H is not singled out in this sequence.
- One might imagine a hypothetical particle with rest-energy $10^6 m_H c^2 \simeq 125$ PeV. This energy scale lies far beyond the reach of contemporary TeV collider experiments, but it may turn out to be relevant to detections of highly energetic cosmic rays in the PeV-EeV range (e.g., [120–122]) and ultra-energetic muons from cosmic or stellar neutrinos [123–125].

Notes

1. By choosing $\hbar$ to scale the phase in rotation operators and to replace Poisson brackets by commutators, Dirac [4] established a consistent description of group-theoretic representations and the Lie algebra in the Hilbert space of the electron. The total angular momentum $\boldsymbol{J}$ generates rotations of the electron wavefunction, and the commutation relations $[J_k, J_\ell] = i\hbar\epsilon_{k\ell m} J_m$ match the Lie algebra of the rotation group SU(2) for half-integer spin (or SO(3) for integer spin). This choice of $\hbar$ ensures that both the infinitesimal generators and the finite rotation operators produce phases consistent with 2π-periodicity, so that the Hilbert space forms a proper representation of the rotation group. However, the use of $\hbar$ beyond these principles of quantum mechanics—for instance, in the definitions of coupling constants and Planck units—is unwarranted and theoretically indefensible, since $\hbar$ carries a geometric imprint of 2-D rotations.
2. Engineers have always used in practice the constant $Z_0 \simeq 377\ \Omega$, all the while failing to recognize that the impedance of free space always appears as $Z_0/(4\pi) \simeq 30\ \Omega$, a value precisely equal to the Planck unit of electric resistance $\mathcal{R}_P = K/c$.
3. The factor of 4π advertises the 3-D geometry of the vacuum. The volume of an n-sphere (or n-ball) is $V_n = (\rho_m/n)S_{n-1}$, where $n = 3$ is the dimension of space in which the surface is embedded and $\rho_m = r$ is the radius of the mean curvature of its surface [17–19]. So, it is the surface area $S_2 = 4\pi r^2$ that brings its 4π tag into the volume V_3. In contrast, in $n = 2$ dimensions, then $V_2 = (r/2)S_1$ for a circle, and the circumference $S_1 = 2\pi r$ brings a two-dimensional tag (viz. 2π) into the area V_2. Thus, the surface of a sphere 'knows' that it lives in 3-D space, and the circumference of a circle 'knows' that it lives in 2-D space. We see then that, despite formally having a dimension of $(n-1)$, the boundaries of these geometric objects are nonetheless aware of the dimension (n) of their enclosed 'content' which of course may be empty (see Ref. [17] for more details).
4. An older idea, proposed before the advent of MOND, to solve the problem of flat rotation curves in spiral galaxies by an ad-hoc $1/r$ force [50–52] did not fare as well. Not noticed at the time, the modification of the force law also changed the source of gravity between small and large scales.
5. Currently espoused ideas about fundamental constants that should be retired:
   (a) *'The Planck mass, M_P, does not look like a particularly good choice* [for a fundamental mass] *since* $\cdots$*'* [25], Part II.3—rebuttal in Section 5 presenting the Planck system reformulated in terms of $M_P = \aleph_A m_e$.
   (b) *'Perhaps one day,* $\cdots$*, we will understand how m_p* [proton mass] *is related to M_P, but in QFT+GR it is not.'* [25], Part II.3—rebuttal in the last row of Table 4, where $M_P = \aleph_A m_e$.
   (c) *'cannot agree that the electron mass or G_F^0 are as good for the role of fundamental unit as the Planck mass or G.'* [25], Part I.7—rebuttal in the last row of Table 4, where $M_P = \aleph_A m_e$, implying that m_e and $\aleph_A$ are fundamental constants.
   (d) '$\cdots$ [fundamental] *constants whose value we cannot calculate with precision in terms of more fundamental constants* $\cdots$ *because we do not know of anything more fundamental.'* [27], Abstract—rebuttal in the notes to Table 4, where in the derivation of

$h = GM_P^2/c$, no constant is fundamental since $M_P = \aleph_A m_e$, $c = 1/\sqrt{\varepsilon_0 \mu_0}$, and $\mathcal{N}(G) = \mathcal{N}\left((k_{B,MJ}/e)^2/(4\pi\varepsilon_0)\right)$. There also is a rational problem with this statement: in a relation between 3 constants, which two are fundamental?

6 We note that the Compton radius r_c is the G-M of the other two atomic radii, the Bohr radius $r_b = r_c/\alpha$ and the classical electron radius $r_e = r_c\alpha$. Hence, it is also the G-M of all three atomic radii, viz. $r_c = \sqrt[3]{r_b r_c r_e}$. It was r_b, however, that was adopted as a fundamental length in the atomic system of units [11], a system which paradoxically does not use c (or G) as a base unit either.

7 The original definition of the Hawking temperature $\Theta_{BH} = \hbar a/(2\pi k_B c)$ [68,71] contains a geometric scaling of $\hbar/(2\pi) \propto 1/(4\pi^2)$ that may be removed since all physical quantities involved are intrinsically 3-D in nature and do not need any geometric imprints. In the original definition of Θ_{BH}, one factor of 2π comes from the unit of $\hbar$ in the action integral that gives the tunneling probability of particles across the horizon, and the other comes from treating the Euclidean time near the horizon as an angular coordinate measured in radians [68,71]. When units of h and cycles are introduced instead, these 2π factors do not appear, and then Equation (24) is obtained (part ③ of Section 4).

8 In a similar vein, the Planck charge $Q_P \propto \sqrt{\varepsilon_0}$ appears to be a minimized charge, but it does not limit the values of e and $Q_\star$ (Section 1.2.6) for which $e < Q_\star < Q_P$. The new charge $Q_\star$ may prove to be just a scale, but the elementary charge is realized physically in protons.

9 Since $\alpha_W = \sqrt{\alpha}$, one may argue that the electroweak interaction has actually only one coupling constant, conventionally taken to be the FSC. Then, the vacuum intervenes to enhance (maximize) this constant (Equation (18)), all the while remaining unaware of the existence of the two extreme couplings $\alpha_g \ll \alpha$ and $\alpha_s \gg \alpha$ that are effectively set by the number of particles in the universe.

10 The Avogadro factor f_A turns out to be larger by 0.239% when determined from the experimental values of the strong coupling constant α_s and the FSC ($f_A = \sqrt{\beta_s}$; see Equation (11) and the notes to Table 10). In this work, we have adopted the slightly lower value determined in Table 4 from $f_A = N_A m_e/M_P$ because these constants (including our 10-SD value of G from Table 2 used to determine $M_P = \sqrt{hc/G}$) are currently measured to much higher precisions than α_s.

11 The fundamental unitless constants given in Section 3 are all expressed as the squares of various quantities or scales. To the extent that squared ratios reflect an underlying natural property, we can ascribe significance to charge $Q_\star$ as follows: the 'weak' squared ratio $(e/Q_\star)^2 = \alpha_W$ corresponds to the EM squared ratio $(e/Q_P)^2 = \alpha$; and the analogy extends to the gravitational sector, where $(m_e/M_\star)^2 = (\aleph_A)^{-1}$ and $(m_e/M_P)^2 = \alpha_g$, respectively. Here, $Q_\star$ is the G-M of e and Q_P, $M_\star$ is the G-M of m_e and M_P, $\alpha_W = \sqrt{\alpha}$, and $(\aleph_A)^{-1} = \sqrt{\alpha_g}$.

12 The coefficient 1.042 962 076 is also realized in Equation (A5) of the Planck voltage for obvious reasons.

13 Equation (A10) relates the FSC α to the Compton radius r_c and the physical properties of the electron (e, m_e). At the Planck scale, the FSC and the weak coupling constant $\alpha_W = \sqrt{\alpha}$ find yet another interpretation expressed by the ratio of Planck energies $(Ke^2/L_P)/(k_B\Theta_P) = \alpha$. This is the first time that the electroweak constants (α, α_W) have been connected to Planck scale properties.

14 We note that the measured boson masses m_H and m_W [31] currently predict the electron's mass-to-charge ratio $\mathcal{N}(m_e/e) = \mathcal{N}(\varepsilon_0)(m_W/m_H)$ to within 0.0333% accuracy.

15 The dimensions assigned to factor 10^{-6} that would convert the relations in Table A3 into identities are $[c/\mathcal{V}]^2[\Theta]$ or, equivalently, $[\Theta]/[\mu_0 \mathcal{I}]^2$. The latter designation effectively combines temperature with magnetic-field circulation [119] via Ampére's law—quite an innovative blending of physical quantities and their units.

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