Convolution Theory

INTRODUCTION

When dealing with dynamic measurements and digital signals, one of the most important mathematical theorems next to the Fourier transformation is the convolution integral. The convolution integral is, in fact, directly related to the Fourier transform, and relies on a mathematical property of it. It is because of this property that makes the convolution integral often very convenient to use and has a wide variety of uses in many different fields and applications.

Before one can start using the convolution integral, it is important to understand it first. In order to make understanding the convolution integral a little easier, this document aims to help the reader by explaining the theorem in detail and giving examples.

BASIC DEFINITION

The basic mathematical definition of convolution is the integral over all space of one function at x multiplied another function at u-x, taken with respect to x. It should be noted that x can represent anything including time, frequency, or even three dimensional space, depending on the application. It should also be noted that the convolution results in a function of a new variable u, which will represent the same domain as the original variable. Convolution is typically represented as a circle with a cross in the center, as follows:

$$f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x) \cdot g(u-x) = \int_{-\infty}^{\infty} g(x) \cdot f(u-x)$$

Another important thing to note is the fact that convolution is commutative, that is, it does not matter which function is taken first.

One popular way of explaining the convolution integral is illustrated in Figure 1. The idea here is to think of the convolution as a weighted summation of shifted copies of one function, where the weighting values for each copy is given by the second function. The first two plots in the figure are the original two functions, g(x) and f(x). The next three pairs of plots show the shifted copies of function g(u-x) on the left and weighted copies on the left, f(x)g(u-x). In theory there would be an infinite number of these copies for all values of x. The bottom left plot shows several shifted and weighted copies of g overlaid on top of one another, to illustrate how they add up. Finally, the bottom right plot is the final convolution of the two original functions.

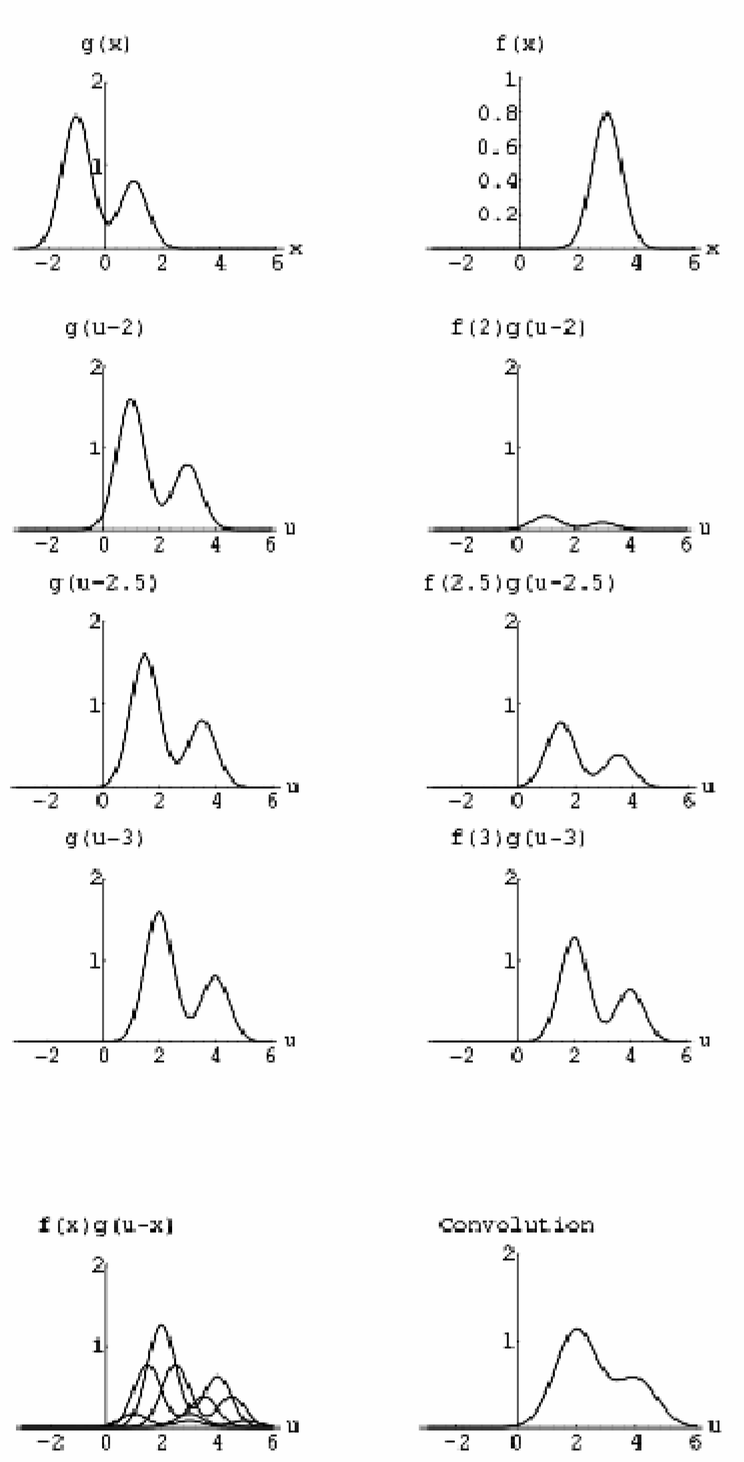

Figure 1: This is a breakdown of the convolution integral. Two signals are given at the top, g(x) and f(x). The function f(x) acts as a weighing function for each value of x that is shifted and copied by g(u-x). The bottom two plots show several weighted and shifted copies of the function g (left) and the final convolution (right). *Source: "The Convolution theorem and its Applications", from the University of Cambridge department of Structural Medicine website:* [*http://www-structmed.cimr.cam.ac.uk/Course/Convolution/convolution.html*](http://www-structmed.cimr.cam.ac.uk/Course/Convolution/convolution.html)*, accessed June 2006.*

The basic mathematical definition by itself may not seem very useful until it has been applied to the Fourier transform. The convolution theorem does this, and can be expressed in two different ways:

The Fourier transform of a convolution is the product of the Fourier transforms

$$\mathscr{F}(f \otimes g) = \mathscr{F}(f) \cdot \mathscr{F}(g)$$

The Fourier transform of a product is the convolution of the Fourier transforms

$$\mathscr{F}(f \cdot g) = \mathscr{F}(f) \otimes \mathscr{F}(g)$$

It should be apparent from these definitions that the convolution of two functions is equivalent to the multiplication of the Fourier transforms of the two functions, and vice versa. For example, when the Fourier transform is applied on time domain signals to obtain frequency spectra, the convolution of the original time domain signals is equal to multiplying the two frequency spectra. This is shown in the following equation:

$$f(t) \otimes g(t) = \mathscr{F}(f(t)) \cdot \mathscr{F}(g(t)) = F(\omega) \cdot G(\omega)$$

PROVING CONVOLUTION THEORY

Before we prove the above statements, let us briefly review the definition of the Fourier transform.

$$\mathscr{F}(f(r)) = F(s) = \int_{-\infty}^{\infty} f(r) \cdot e^{i \cdot s \cdot r}\, dr$$

Note that for using Fourier to transform from the time domain into the frequency domain r is time, t, and s is frequency, ω. This gives us the familiar equation:

$$\mathscr{F}(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{i \cdot \omega \cdot t}\, dt$$

Now to prove the first statement of the convolution theorem; that the Fourier transform of the convolution is the product of the individual Fourier transforms. To do this we start by using the definitions to find the Fourier transform of the convolution.

$$\mathscr{F}\left(f(x)\otimes g(x)\right)=\mathscr{F}\left(\int_{-\infty}^{\infty} f(x)\cdot g(u-x)dx\right)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\cdot g(u-x)dx\cdot e^{i\cdot s\cdot u}\,du$$

Now, in order to separate the two variables, let $w = u - x \rightarrow u = x + w$. Since the limits of integration are infinite, they will not change with the variable. By expanding the exponential, the variables can be easily separated.

$$\mathscr{F}\left(f(x)\otimes g(x)\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\cdot g(w)\cdot e^{i\cdot s\cdot(x+w)}dx\,dw$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\cdot e^{i\cdot s\cdot x}\cdot g(w)\cdot e^{i\cdot s\cdot w}dx\;dw$$

Since the variables of integration have now been separated the integrals themselves can also be separated.

$$\mathscr{F}\left(f(x)\otimes g(x)\right)=\int_{-\infty}^{\infty} f(x)\cdot e^{i\cdot s\cdot x}dx\cdot\int_{-\infty}^{\infty} g(w)\cdot e^{i\cdot s\cdot w}dw$$

Keeping in mind that x can be anything we choose, w can be replaced with x since they both represent the same domain. The result of this substitution makes it fairly easy to recognize the two Fourier transforms.

$$\mathscr{F}\left(f(x)\otimes g(x)\right)=\int_{-\infty}^{\infty} f(x)\cdot e^{i\cdot s\cdot x}dx\cdot\int_{-\infty}^{\infty} g(x)\cdot e^{i\cdot s\cdot x}dx$$

$$=\mathscr{F}\left(f(x)\right)\cdot\mathscr{F}\left(g(x)\right)$$

Having proved the first form of the convolution theorem, rearrange it to derive the second is fairly simple. By starting with the original statement, and taking the inverse Fourier transform of both sides,

$$\mathscr{F}(f\otimes g)=\mathscr{F}(f)\cdot\mathscr{F}(g)$$
$$\mathscr{F}\left[\mathscr{F}^{-1}(F)\otimes\mathscr{F}^{-1}(G)\right]=F\cdot G$$
$$\mathscr{F}^{-1}(F)\otimes\mathscr{F}^{-1}(G)=\mathscr{F}^{-1}(F\cdot G)$$
$$\mathscr{F}(f\cdot g)=\mathscr{F}(f)\otimes\mathscr{F}(g)$$

UNIT IMPULSE RESPONSE AND GENERAL FORCING FUNCTIONS

Consider a single degree of freedom system shown in Figure 2 (a), which is subjected to a unit impulse at t = 0, as shown in Figure 2 (b). The solution of the free vibration motion equation is given as:

$$x(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left[x_0 \cdot \cos(\omega_d \cdot t) + \frac{\dot{x}_0 + \zeta \cdot \omega_n \cdot x_0}{\omega_d} \cdot \sin(\omega_d \cdot t) \right]$$

$$\zeta = \frac{c}{2 \cdot m \cdot \omega_n}$$

Where,
$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

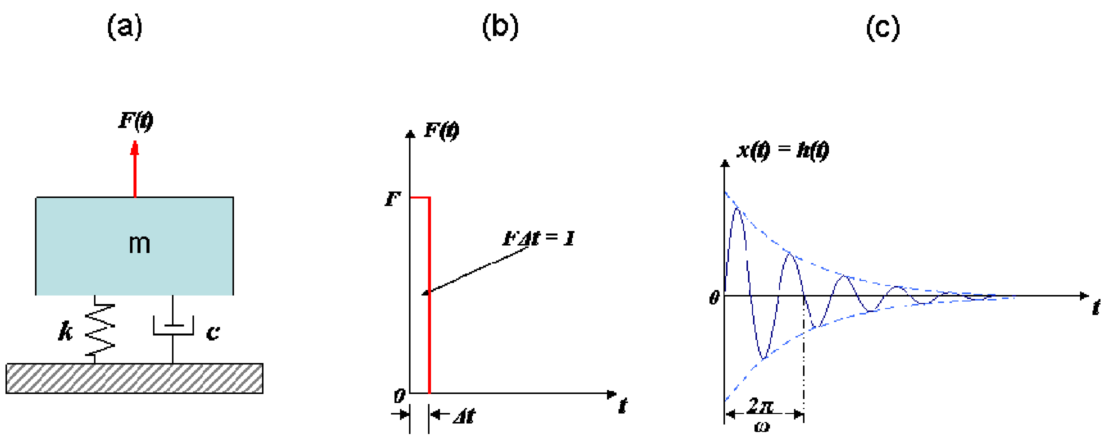

Figure 2: The unit impulse response function is demonstrated here for a single degree od freedom system shown in (a). The impulse force is shown in (b). Finally, the resulting response function is shown in (c). This response function is represented as *h(t)*.

Assuming that the mass is at rest at $t = 0$ then the following conclusions can be made about the intitial conditions of the system:

$$x_0 = 0$$

$$F\Delta t = 1 = m \cdot \dot{x}_0 \rightarrow \dot{x}_0 = \frac{1}{m} \qquad \text{(impulse-momentum relation)}$$

This reduces the above solution to the equation of motion to the following:

$$x(t) = \frac{e^{-\zeta \cdot \omega_n \cdot t}}{m \cdot \omega_d} \cdot \sin(\omega_d \cdot t) = h(t)$$

This equation is known as the impulse response function of the system, and is denoted as *h(t)*. If the magnitude of the impulse is an arbitrary value $\tilde{F}$ instead of unity, then the initial velocity becomes $\tilde{F}/m$, and the response of the system becomes

$$x(t) = \frac{\tilde{F} \cdot e^{-\zeta \cdot \omega_n \cdot t}}{m \cdot \omega_d} \cdot \sin(\omega_d \cdot t) = \tilde{F} \cdot h(t)$$

When the impulse $\tilde{F}$ is applied at an arbitrary time, $t = \tau$, as it is in Figure 3 (a), the change in velocity will still be $\tilde{F}/m$. If the velocity before the impulse is applied zero, then the response will look exactly the same as before, except shifted by τ. This is illustrated in Figure 3 (b), and can be expressed simply as:

$$x(t) = \tilde{F} \cdot g(t - \tau)$$

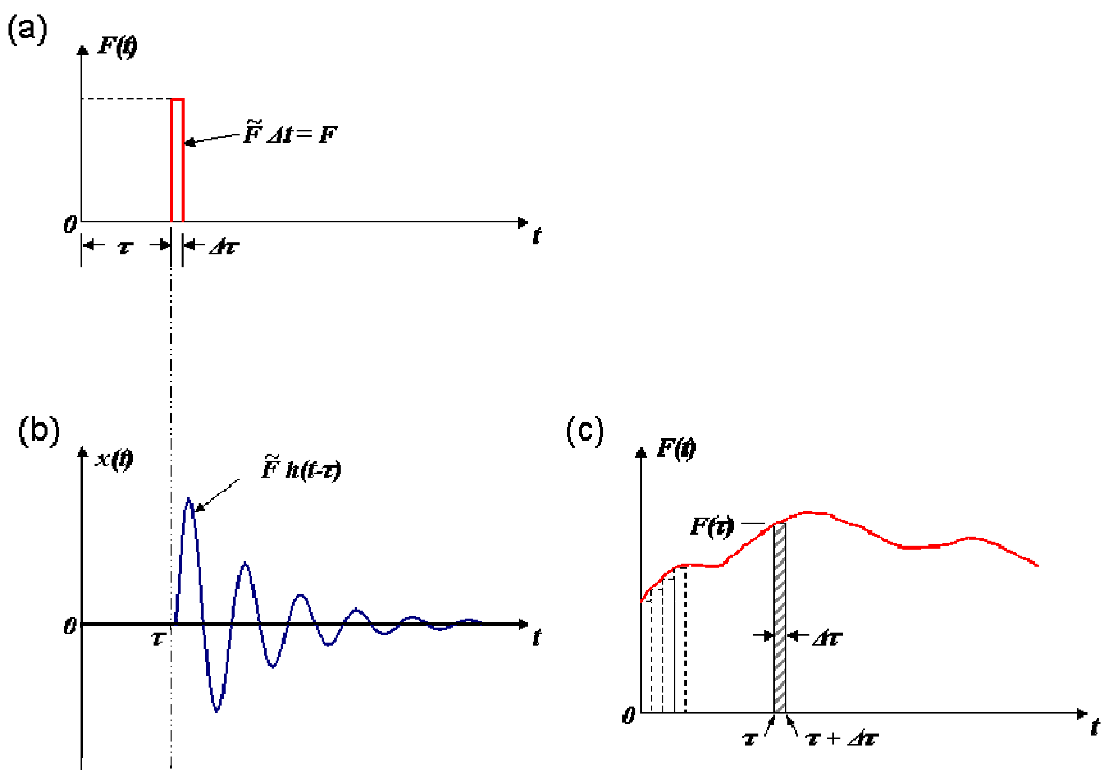

Figure 3: Demonstration of system response to the general forcing condition. When the impulse force occurs at time τ, and is of magnitude $\tilde{F}$, as shown in (a), the response function is shifted and magnified by the same amount, as shown in (b). The plot in (c) demonstrates how an arbitrary forcing function may be taken as an infinite number of impulse functions.

Finally, when the force becomes a completely arbitrary function, *F(t)*, it can be represented as an infinite series of impulses, as shown in Figure 3 (c). If you haven't guessed by now, the total system response is nothing more that the summation of the responses to all these impulses, and an infinite summation can be expressed as an integral:

$$x(t) = \sum_{\infty} F(\tau) \cdot h(t-\tau)\Delta\tau$$

$$x(t) = \int_{-\infty}^{\infty} F(\tau) \cdot h(t-\tau)d\tau$$

This is nothing more than the convolution of the forcing function with the impulse response function. Thus we see that the response function can also be represented as

$$x(t) = h(t) \otimes f(t)$$

Incidentally, the frequency response function H(w) is nothing more than the Fourier transform of the impulse response function, and in the frequency domain, the response spectra to a known system can be found with this equation:

$$X(\omega) = H(\omega) \cdot F(\omega)$$